# LOCAL STABILITY IN A MINIMIZATION PROBLEM FOR CONDUCTIVITY IMAGING


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ABSTRACT. We consider the problem of minimization of the functional $\int_\Omega a(x)|\nabla u(x)|dx$ over functions $u$ of bounded variation with prescribed trace $f$ at the boundary. The stability of the minimum value of the functional with respect to the coefficient $a \in L^2(\Omega)$ is established in the vicinity of a coefficient of the form $a = \sigma|\nabla u|$, where $u$ solves $\nabla \cdot \sigma \nabla u = 0$ with $u|_{\partial\Omega} = f$. This problem occurs in conductivity imaging when knowledge of the magnitude of the current density field inside a body is available. The method of proof is constructive.



**Keywords** Conductivity imaging; Non-smooth optimization; Functions of bounded variation; Degenerate elliptic equations; Regularization.

**Mathematics Subject Classification** 35R30; 49A45.


## 1. Introduction

In this paper we consider a question of stability in a non-smooth minimization problem occurring in the coefficient identification problem of conductivity imaging. The conductive body $\Omega$ is a simply connected domain in $\mathbb{R}^d$, $d \geq 2$, with Lipschitz boundary. Given the electric conductivity $\sigma \in L^\infty(\Omega)$ with $\sigma \geq \delta > 0$ a.e., and the the voltage potential $f \in H^{1/2}(\partial\Omega)$ at the boundary, there is a unique $u \in H^1(\Omega)$, the electric voltage potential, solution to the conductivity equation

$$\nabla \cdot \sigma \nabla u(x) = 0, \quad x \in \Omega, \tag{1}$$

subject to the boundary condition

$$u(x) = f(x) \quad x \in \partial\Omega. \tag{2}$$


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We refer to the solutions of (1) as *σ-harmonic* maps. The electric field is defined by $\nabla u$ and the current density field is given from Ohm's law by $J = \sigma\nabla u$.

The engineering work in [14] presents a modality for determining the current density field $J$ inside $\Omega$ from Magnetic Resonance measurements and two rotations of the object. This raises the question of recovering the conductivity $\sigma$ by using the knowledge of $J$. If two transversal current density fields $J_1, J_2$ are known, then $\sigma$ can be determined via a local formula, see [11, 13]. If only their magnitudes $|J_1|$ and $|J_2|$ are known, then an algorithm for recovering $\sigma$ is available in [5, 12]. To limit the number of experiments, one considers the question of conductivity determination by using just one current density field. In [5] the problem of determining $\sigma$ from knowledge of $|J|$ and the corresponding injected current $\sigma\frac{\partial u}{\partial \nu}$ at the boundary has been shown to have multiple solutions.

The above considerations motivate taking into account the boundary voltage $f$. Throughout this paper $u_0 \in H^1(\Omega)$ denotes the unique solution of (1) and (2).

In two dimensions, knowledge of $|J|$ together with boundary voltage $f$ and the current $\sigma\frac{\partial u_0}{\partial \nu}\big|_{\partial\Omega}$ on some arc of the boundary recovers the conductivity in a region inside well defined by the arc, see [7]. The key fact (valid in dimensions $d \geq 2$) is that the equipotential surfaces (the level sets of $u_0$) are simultaneously characteristics of the degenerate elliptic equation $\nabla \cdot |J||\nabla u|^{-1}\nabla u = 0$ and minimal surfaces (geodesics if $d = 2$) in the metric $g = |J|^{d/(d-1)}I$ conformal with the Euclidean metric. By determining the minimal surfaces, one determines the level sets of $u_0$, and thus the voltage potential itself. Outside the set of singular points of $u_0$, the conductivity can be recovered via $\sigma = |J|/|\nabla u_0|$. In two dimensions there are known conditions on the boundary voltage $f$ that insure a non-vanishing gradient throughout $\Omega$, see [7]. In three or higher dimensions identifying the boundary voltage $f$ such that the voltages $u_0$ is free of singular points is still an open question.

Another relation connecting $|J|, f$ and $u_0$ has been established in [8]. Namely, the voltage potential $u_0$ is one of the minimizers of the problem:

$$\min\left\{\int_\Omega |J|\ |\nabla u| dx :\ u \in H^1(\Omega),\ u|_{\partial\Omega} = f\right\}. \tag{3}$$

Moreover, $u_0$ is unique in the class of $W^{1,1}(\Omega)\bigcap C(\overline{\Omega})$ minimizers with a negligible set of singular points ($|\nabla u| > 0$, a.e. $\Omega$), see [8] for details.

In this paper, we study the continuous dependence on $|J| \in L^2(\Omega)$ in the minimization of the functional in (3) over the larger space $BV(\Omega)$ of functions of bounded variation. The boundary data $f \in H^{1/2}(\partial\Omega)$ is assumed to be exact. This assumption agrees with the practical experiment in which only $|J|$ is measured while the voltage pattern $f$ is maintained by the apparatus. For convenience, the magnitude of the current density field will be denoted by $a$, i.e. $a = |J|$. With these notations the functional in (3) becomes

$$F[h; a] = \int_\Omega a(x)|\nabla u_0(x) + \nabla h(x)|dx \tag{4}$$

and we study the minimization of $F$ over $h \in BV_0(\Omega)$, i.e. $h \in BV(\Omega)$ with vanishing trace (in the sense of traces of functions of bounded variations) at the boundary. A detailed explanation on the appropriateness of the sample space together with the difficulties arising by this choice is presented in Section 2.

Under the additional regularity assumptions that $u_0 \in C^1(\Omega)$ and $\sigma \in C(\Omega)$, the map $h = 0$ is a minimizer of (4) in the larger space $BV(\Omega)$; see Proposition 4.1 below. Such further regularity can be guaranteed a priori for sufficiently regular conductivities and boundary voltage from the elliptic regularity up to the boundary results; see the concluding remarks of Section 6.

Following [8], a pair $(a, f) \in L^2(\Omega) \times H^{1/2}(\partial\Omega)$ is called *admissible* if there is a map $\sigma \in L^\infty(\Omega)$, the *generating conductivity*, and the $\sigma$-harmonic map $u$, the so-called *generating potential*, in $\Omega$ with $u = f$ on $\partial\Omega$ and such that $a = \sigma|\nabla u|$.

Let $a \in L^2(\Omega)$ such that $(a, f)$ is admissible. In Theorem 5.1 we show the continuous dependence of the minimum value of the functional (4) with respect to perturbation in $L^2(\Omega)$. The proof is based on the regularization methods developed for the minimization of non-smooth functionals in nonreflexive Banach spaces in [9, 10]. The proof also provides a stable minimization scheme.

Throughout the paper we use the following notations for the norms: $\|a\|$ denotes the $L^2(\Omega)$-norm, $\|h\|_{1,0} = \left(\int_\Omega |\nabla h|^2 dx\right)^{1/2}$ denotes the $H^1_0(\Omega)$-norm, $\|h\|_{1,1} = \int_\Omega |\nabla h| dx$ denotes the $W^{1,1}_0$-norm and $\|f\|_{1/2}$ denotes the $H^{1/2}(\partial\Omega)$-norm. For $u \in BV(\Omega)$ we denote by $\|Du\|$ the positive Radon measure defined on any open set $U \subseteq \Omega$ by

$$\|Du\|(U) = \sup\{\int_\Omega u\nabla \cdot f : \; f = (f_1, ..., f_d) \in C^1_0(U; \mathbb{R}^d), \; |f| \leq 1\},$$

where $|f| = \sqrt{f_1^2 + ...f_d^2}$.

## 2. Non-smooth minimization problems in non-reflexive Banach spaces

In this section we motivate the appropriateness of $BV(\Omega)$ as the sample space. We start by presenting some general considerations from the theory of the calculus of variations to emphasize the difficulties in the minimization problem generated by the non-reflexivity of $BV(\Omega)$.

The traditional method in the calculus of variations used in proving existence of a minimum of a functional is to establish precompactness of minimizing sequences and the lower semicontinuity of the functional. While it is true that a lower semi-continuous functional on a compact set $S$ in a topological space attains its minimum in $S$, the norm topology is not appropriate since a closed ball is not compact in this norm. This motivates endowing a Banach space with a topology relative to which the set $S$ becomes compact. In reflexive Banach spaces the weak topology accomplishes this task. The role of the weak topology is derived from two important results: the Alaoglu-Bourbaki-Kakutani's theorem and the Mazur's theorem. The first theorem states that the closed ball in any dual space $B^*$ is compact in the weak-star topology of $B^*$. For reflexive spaces, the weak topology of $B$ coincide with the weak-star topology of $B^{**}$, and thus the ball is weakly compact. The second theorem, due to Mazur, states that every strongly closed convex set in a Banach space is weakly closed; hence in a reflexive Banach space every bounded closed convex set is weakly compact. On the basis of these results the existence of a minimizer follows: *If $f$ is a weakly lower semicontinuous real valued function on a weakly closed subset $S$ of a reflexive Banach space $B$, then $f$ attains its infimum in $S$.* In particular $S$ can be a bounded, closed and convex subset of $B$. The utility of this theorem depends on establishing sufficient conditions for $f$ to be weakly lower semicontinuous. Often a coercivity condition or some assumptions on derivatives are imposed to guarantee the weak lower semicontinuity. For example, if $f$ has continuous second order Gateaux differential $D^2f(x;h,h)$ on all of $B$ and if $D^2f(x;h,h) \geq \|h\|\gamma(\|h\|)$, where $\gamma(t)$ is a nonnegative continuous function with $\lim_{t\to\infty}\gamma(t) = \infty$, then $f$ attains its infimum on $B$. If $\gamma(t) > 0$ then the minimizer is unique.

Optimization theory (existence, uniqueness, and stability) in *nonreflexive* Banach spaces is more subtle than the theory in reflexive Banach spaces, partly due to the fact that the weak topology of a nonreflexive Banach space does not coincide with the weak-star topology of its dual. In particular, the theorems of Alaoglu-Bourbaki-Kakutani and Mazur have no direct relevance.

The choice of the sample space $BV(\Omega)$ is motivated by the following observation. Consider for simplicity the case when $a \equiv 1$. If $\{u_n\} \subset W^{1,1}(\Omega)$ is a minimizing sequence for $\int_\Omega |\nabla u| dx$ such that $u_n \to \underline{u}$ in $L^1_{loc}(\Omega)$, then it is the case that $\underline{u} \in BV(\Omega)$ but $\underline{u}$ may not necessarily lie in $W^{1,1}(\Omega)$. Moreover, it is the total variation of the gradient measure $\|Du_n\|$ which is lower semicontinuous with respect to the $L^1_{loc}$ convergence (albeit for $u_n \in W^{1,1}(\Omega)$, $\|Du_n\|(\Omega) = \int_\Omega |\nabla u_n| dx$), see e.g. [16, Theorem 5.2.1].

Another marked difference arising in the consideration of stability of the minimizing sequences concerns the approximation of functions in $BV(\Omega)$. In general, if $u$ is merely in $BV(\Omega)$ (and not smoother) it cannot be expected to find a sequence of functions $u_n \in C^\infty(\Omega)$ such that $u_n \to u$ in $L^1(\Omega)$ and $\int_\Omega |D(u_n - u)| \to 0$. This is due to the fact that the closure of $C^\infty$-functions with respect to the $BV$-norm is the Sobolev space $W^{1,1}(\Omega) \subsetneq BV(\Omega)$. For example, a function such as piecewise constant (which is in $BV(\Omega) \setminus W^{1,1}(\Omega)$) cannot be approximated by a family of smooth functions. Implications of this situation in connection with the conductivity imaging problem are analyzed in Sections 4 and 5.

Existence and uniqueness of solutions of various classes of optimization problems in nonreflexive Banach spaces have been extensively studied in literature. However, works on stability with respect to the data are not so common. The usual path of working with weaker norms is not always sufficient for establishing a stability result. Such is the situation in the imaging conductivity problem of concern here. More precisely, although the coefficient $a$ is admissible and, thus, the existence of a minimizer is nearly postulated (see Proposition 4.1), the nearby coefficients may not be admissible and the existence of the minimizers for the perturbed problem is not guaranteed. This is where the ill-posedness shows in our problem. To circumvent the ill-posedness we use a regularization approach introduced in [9, 10]. The idea is to consider a family of modified functionals on the Hilbert space $H^1_0(\Omega)$ which is appropriately embedded in the nonreflexive Banach space $BV(\Omega)$, and to consider the minimizers of the modified functionals as approximate solutions of the original problem. This approach is presented in Section 5.

## 3. A regularized minimization problem

Let $a \in L^2(\Omega)$ with $a \geq 0$ and $f \in H^{1/2}(\partial\Omega)$. The results in this section do not require $(a, f)$ to be admissible.

Since $u_0$ is the $\sigma$- harmonic map with trace $f$ on the boundary, the elliptic regularity up to the boundary, see e.g. [2], yields

$$\|\nabla u_0\| = \left(\int_\Omega |\nabla u_0|^2 dx\right)^{1/2} \leq \|u_0\|_{H^1(\Omega)} \leq C\|f\|_{1/2}, \tag{5}$$

for a constant $C$ depending on the $\delta := ess\ inf(\sigma) > 0$ and $\Omega$.

As explain previously, for an arbitrary $\tilde{a} \in L^2(\Omega)$ the functional $h \mapsto F[h; \tilde{a}]$ may not have a minimizer in $H_0^1(\Omega)$, $W_0^{1,1}(\Omega)$ or $BV_0(\Omega)$. We regularize the functional such that the new functional has a unique minimizer in $H_0^1(\Omega)$. More precisely, for $\epsilon > 0$ arbitrarily fixed, define the following regularization of $F$:

$$\begin{aligned} F_\epsilon[h; a] &:= \int_\Omega a|\nabla u_0 + \nabla h| dx + \epsilon \int_\Omega |\nabla h|^2 dx \\ &= F[h; a] + \epsilon\|h\|_{1,0}^2. \end{aligned} \tag{6}$$

The following lemma shows that $h \mapsto F_\epsilon[h; a]$ is weakly lower semi-continuous on $H_0^1(\Omega)$.

**Lemma 3.1.** *Assume $a \in L^2(\Omega)$. Let $\{h_n\}$ be a sequence in $H_0^1(\Omega)$ and $h \in H_0^1(\Omega)$ with $h_n \rightharpoonup h$ in $H_0^1(\Omega)$. Then*

$$F_\epsilon[h; a] \leq \liminf_{n\to\infty} F_\epsilon[h_n; a]. \tag{7}$$

*Proof.* Let $\{a_m\}$ be an increasing sequence in $C(\Omega) \bigcap L^2(\Omega)$ which converges (in $L^2(\Omega)$ sense) to $a$.

For each fixed index $m$ let $f = (f_1, ..., f_d) \in C_0^1(\Omega; \mathbb{R}^d)$ be arbitrary with $|f| \leq a_m$. Since $h_n \rightharpoonup h$ in $L^2(\Omega)$ we have

$$\begin{aligned} \int_\Omega (h + u_0)\nabla \cdot f dx &= \lim_{n\to\infty} \int_\Omega (h_n + u_0)\nabla \cdot f dx \\ &\leq \liminf_{n\to\infty} \left(\sup\left\{\int_\Omega (h_n + u_0)\nabla \cdot f dx : \ f \in C_0^1(\Omega; \mathbb{R}^d), \ |f| \leq a_m\right\}\right) \\ &= \liminf_{n\to\infty} \int_\Omega a_m|\nabla(h_n + u_0)| dx \leq \liminf_{n\to\infty} \int_\Omega a|\nabla(h_n + u_0)| dx. \end{aligned} \tag{8}$$

The last inequality above uses the fact that $a_m \leq a$.

By taking the supremum in (8) over all $f \in C_0^1(\Omega; \mathbb{R}^d)$ with $|f| \leq a_m$ we get

$$\begin{aligned} \int_\Omega a_m|\nabla(h + u_0)| dx &= \sup\left\{\int_\Omega (h + u_0)\nabla \cdot f dx : \ f \in C_0^1(\Omega; \mathbb{R}^d), \ |f| \leq a_m\right\} \\ &\leq \liminf_{n\to\infty} \int_\Omega a|\nabla(h_n + u_0)| dx. \end{aligned} \tag{9}$$

By letting $m \to \infty$ in (9) we obtain the lower semicontinuity for $F[h;a]$, the first term in (6).

The lower semicontinuity of the second term in (6) is a classical result in the theory of calculus of variations which follows from Fatou's lemma applied to

$$\|h_n\|_{1,0}^2 - \|h\|_{1,0}^2 \geq \int_\Omega \nabla h \cdot \nabla (h_n - h) dx, \ \forall n \in \mathbb{N}.$$

□

Since $h \mapsto F_\epsilon[h;a]$ is coercive in $H_0^1(\Omega)$ (by the regularization) and strictly convex, $F_\epsilon$ has a unique minimizer in $H_0^1(\Omega)$, say

$$h_\epsilon := \operatorname{argmin} \left\{ F_\epsilon[h;a] \ : h \in H_0^1(\Omega) \right\}. \tag{10}$$

The next result shows the continuous dependence of the minimizer on the weight $a \in L^2(\Omega)$.

**Theorem 3.2.** *Let $\epsilon > 0$ be fixed. Let $\{a_n\}$ be an $L^2(\Omega)$-sequence convergent to $a$ in $L^2(\Omega)$, and $\{h_{\epsilon,n}\}$ be the corresponding minimizing sequence of $F_\epsilon[\cdot\ ; a_n]$; i.e., $h_{\epsilon,n} :=$ argmin$\{F_\epsilon[h;a_n];\ h \in H_0^1(\Omega)\}$.*

*Then $h_{\epsilon,n} \rightharpoonup h_\epsilon$ in $H_0^1(\Omega)$ with $n \to \infty$. In particular $h_{\epsilon,n} \to h_\epsilon$ in $L^q(\Omega)$ for $1 \leq q < 2d/(d-2)$.*

*Proof.* Let $\delta_n = \|a - a_n\|$ and consider $n$ large enough so that

$$\delta_n \leq \|a\|. \tag{11}$$

Note first that the sequence $\{h_{\epsilon,n}\}$ is bounded in $H_0^1(\Omega)$. Indeed,

$$\begin{aligned} \epsilon \|h_{\epsilon,n}\|_{1,0}^2 \leq & F_\epsilon[h_{\epsilon,n}; a_n] \leq F_\epsilon[0;a_n] = F[0;a] + F[0;a_n - a] \\ \leq & (\|a\| + \delta_n)\|\nabla u_0\| \leq 2C\|a\|\ \|f\|_{1/2} \end{aligned}$$

The second inequality uses the minimizing property defining $h_{\epsilon,n}$, and the last one uses (5) and (11). Thus

$$\|h_{\epsilon,n}\|_{1,0} \leq \frac{\sqrt{2\|a\|C\|f\|_{1/2}}}{\sqrt{\epsilon}} =: C_\epsilon \tag{12}$$

and the sequence $\{h_{\epsilon,n}\}$ has an $H_0^1$- weakly convergent subsequence; say $h_{\epsilon,n_k} \rightharpoonup \tilde{h}_\epsilon$, for some $\tilde{h}_\epsilon \in H_0^1(\Omega)$. Note that $C_\epsilon = O(1/\sqrt{\epsilon})$ blows up as $\epsilon \downarrow 0$.

Let $h \in H_0^1(\Omega)$ be an arbitrary map and $C_\epsilon$ be as defined in (12). Since the map $h \mapsto F_\epsilon[h;a]$ is weakly lower semi-continuous on $H_0^1(\Omega)$, the following estimates hold:

$$\begin{aligned}
F_\epsilon[\tilde{h}_\epsilon; a] \leq & \liminf_{k\to\infty} F_\epsilon[h_{\epsilon,n_h}; a] = \liminf_{k\to\infty} \{F_\epsilon[h_{\epsilon,n_h}; a_{n_k}] + F[h_{\epsilon,n_h}; a - a_{n_k}]\} \\
\leq & \liminf_{k\to\infty} \{F_\epsilon[h_{\epsilon,n_h}; a_{n_k}] + \delta_{n_k}(\|h_{\epsilon,n_h}\|_{1,0} + \|\nabla u_0\|)\} \\
\leq & \liminf_{k\to\infty} \left\{F_\epsilon[h_{\epsilon,n_h}; a_{n_k}] + \delta_{n_k}(C_\epsilon + C\|f\|_{1/2})\right\} \\
\leq & \liminf_{k\to\infty} \left\{F_\epsilon[h; a_{n_k}] + \delta_{n_k}(C_\epsilon + C\|f\|_{1/2})\right\} \\
= & \liminf_{k\to\infty} \left\{F_\epsilon[h; a] + F[h; a_{n_k} - a] + \delta_{n_k}(C_\epsilon + C\|f\|_{1/2})\right\} \\
\leq & \liminf_{k\to\infty} \left\{F_\epsilon[h; a] + \delta_{n_k}(\|h\|_{1,0} + C_\epsilon + C\|f\|_{1/2})\right\} \\
\leq & \limsup_{k\to\infty} \left\{F_\epsilon[h; a] + \delta_{n_k}(\|h\|_{1,0} + C_\epsilon + C\|f\|_{1/2})\right\} \\
\leq & F_\epsilon[h; a] + \limsup_{k\to\infty} \delta_{n_k}(\|h\|_{1,0} + C_\epsilon + C\|f\|_{1/2}) \\
= & F_\epsilon[h; a].
\end{aligned}$$

In the estimate above the third inequality uses (5) and (12), while the fourth inequality uses the minimizing property defining $\{h_{\epsilon,n_k}\}$.

Therefore $\tilde{h}_\epsilon$ is a minimizer for $F_\epsilon$ in $H^1_0(\Omega)$. Since the minimizer is unique, $\tilde{h}_\epsilon = h_\epsilon$. Since any other weakly convergent subsequence of $\{h_{\epsilon,n}\}$ also converges to $h_\epsilon$, the entire sequence is weakly convergent to $h_\epsilon$. An application of the Rellich-Kondrachov compactness imbedding yields the strong convergence in $L^q(\Omega)$, for $1 \leq q < 2d/(d-2)$; see e.g., [3]. □

## 4. A global minimization property in $BV(\Omega)$

For $u \in BV(\Omega)$ the $\|Du\|$ is a Radon measure, which, by Riesz representation theorem, can be thought of as a positive linear functional on $C(\Omega)$. Moreover, if $C_+(\Omega)$ denotes the set of nonnegative continuous maps on $\Omega$, for any $g \in C_+(\Omega)$, we also have

$$\|Du\|(g) = \sup\left\{\int_\Omega u\nabla\cdot f dx : \ f \in C^1_0(\Omega;\mathbb{R}^n), \ |f| \leq g\right\};$$

see, e.g., [16].

The following result extends the minimizing property in [8, Proposition 1.2] to the space of $BV(\Omega)$.

**Proposition 4.1.** *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be an open domain with Lipschitz boundary. Assume that $\sigma \in L^\infty(\Omega)$ and $u_0 \in H^1(\Omega)$ is $\sigma$-harmonic satisfying $\sigma|\nabla u_0| \in C(\Omega)$. Then*

(13) $$u_0 \in \mathit{argmin}\{\|Du\|(\sigma|\nabla u_0|) : \ u \in BV(\Omega), \ u|_{\partial\Omega} = u_0|_{\partial\Omega}\}.$$

*Proof.* In [8, Proposition 1.2] it is shown that

$$u_0 \in \operatorname{argmin}\{\int_\Omega \sigma|\nabla u_0|\ |\nabla u|dx:\ u \in W^{1,1}\Omega),\ u|_{\partial\Omega} = u_0|_{\partial\Omega}\}. \tag{14}$$

However, for $u \in W^{1,1}(\Omega)$, $\int_\Omega \sigma|\nabla u_0|\ |\nabla u|dx = \|Du\|(\sigma|\nabla u_0|)$.

By the density property of smooth functions in $BV(\Omega)$, see e.g. [1, 4], for each $u \in BV(\Omega)$ there is a sequence $\{u_n\}$ in $C^\infty(\Omega)$ such that:

(i) $u_n \to u$ in $L^1(\Omega)$,

(ii) $\|Du_n\| \rightharpoonup \|Du\|$,

(iii) $u_n|_{\partial\Omega} = u|_{\partial\Omega}$ as traces of functions of bounded variation.

The weak convergence in (ii) is in the sense of Radon measures, in particular, for any $a \in C_+(\Omega)$,

$$\int_\Omega a(x)|\nabla u_n|dx = \|Du_n\|(a) \to \|Du\|(a). \tag{15}$$

Thus, for all $n \in \mathbb{N}$,

$$\int_\Omega \sigma|\nabla u_0|^2 dx \leq \int_\Omega \sigma|\nabla u_0|\ |\nabla u_n|dx = \|Du_n\|(\sigma|\nabla u_0|). \tag{16}$$

By taking the limit in the inequality above and by using (15) the conclusion follows ☐

The result below is a direct consequence of the definition of admissibility and Proposition 4.1. By $BV_0(\Omega)$ we denote the (Banach) subspace of functions of bounded variation with vanishing trace at the boundary.

**Corollary 4.2.** *Assume that $(a, f) \in L^2(\Omega) \times H^{1/2}(\partial\Omega)$ is an admissible pair such that $a \in C(\Omega)$. Then*

$$0 \in \mathit{argmin}\{\|D(u_0 + h)\|(a):\ h \in BV_0(\Omega)\}. \tag{17}$$

Some sufficient conditions for the regularity of $a$ are briefly mentioned in Section 6.

## 5. Convergence of the regularized minimizers with $\epsilon \downarrow 0$

We start with the admissible pair $(a, f) \in L^2(\Omega) \times H^{1/2}(\partial\Omega)$, say generated by some conductivity $\sigma \in L^\infty(\Omega)$, and $u_0 \in H^1(\Omega)$ be the corresponding potential. Further regularity on $a$ will be imposed as needed.

The equation (14) shows that, for all $h \in H^1_0(\Omega)$,

$$F[0; a] \leq F[h; a]. \tag{18}$$

Let $\{a_n\}$ be a sequence convergent to $a$ in $L^2(\Omega)$ and $\epsilon_n \downarrow 0$. Let $\{h_{\epsilon_n}\}$ be the sequence of the minimizers corresponding to the functionals $F_{\epsilon_n}[u; a_n]$, i.e.

$$h_{\epsilon_n} := \operatorname{argmin}\{F_{\epsilon_n}[h; a_n] : \; h \in H_0^1(\Omega)\}. \tag{19}$$

In Section 3 we showed that $\{h_{\epsilon_n}\}$ is well-defined. Note that both the regularized parameter and the weight of the functional are changing with $n$. We show below that if the regularized parameter $\epsilon_n$ is chosen so that $\|a - a_n\| = o(\sqrt{\epsilon_n})$, then $\{h_{\epsilon_n}\}$ is a minimizing sequence for $F$ in (4). Moreover, we identify sufficient conditions for the sequence $\{h_{\epsilon_n}\}$ to converge to 0.

**Theorem 5.1.** *Assume that $(a, f) \in L^2(\Omega) \times H^{1/2}(\partial\Omega)$ is admissible, and let $u_0 \in H^1(\Omega)$ be the corresponding potential. Let $\{a_n\}$ be a sequence in $L^2(\Omega)$ with $a_n \to a$ in $L^2(\Omega)$, and let $\epsilon_n \downarrow 0$ in such a way that*

$$\lim_{n\to\infty} \frac{\|a - a_n\|^2}{\epsilon_n} = 0. \tag{20}$$

*Let $\{h_{\epsilon_n}\}$ be the sequence of the minimizers of $F_{\epsilon_n}[\cdot\ ; a_n]$ as in* (19). *Then*

$$\liminf_{n\to\infty} F_{\epsilon_n}[h_{\epsilon_n}; a_n] = \liminf_{n\to\infty} F[h_{\epsilon_n}; a_n] = F[0; a]. \tag{21}$$

*If, in addition, $a \in C(\Omega)$ and*

$$\inf_{\overline{\Omega}} a =: \alpha > 0, \tag{22}$$

*then $\{h_{\epsilon_n}\}$ has a subsequence which is convergent in $L^q(\Omega)$ to some $\underline{h}$ for $1 \le q < \frac{d}{d-1}$. Moreover $\underline{h} \in BV_0(\Omega)$ and*

$$\|D(u_0 + \underline{h})\|(a) = min\{\|D(u_0 + h)\|(a) : \; h \in BV_0(\Omega)\} = F[0; a]. \tag{23}$$

*Proof.* Let $\delta_n = \|a - a_n\|$ be as before.

Since $h_{\epsilon_n} \in H_0^1(\Omega)$ and 0 is a minimizer of $F[\ \cdot\ ; a]$ over $H_0^1(\Omega)$ we have

$$F[0; a] \le \liminf_{n\to\infty} F[h_{\epsilon_n}; a] \le \liminf_{n\to\infty} F_{\epsilon_n}[h_{\epsilon_n}; a]. \tag{24}$$

We claim that the reverse inequality also holds. Indeed,

$$
\begin{aligned}
\liminf_{n\to\infty} F_{\epsilon_n}[h_{\epsilon_n};a] &= \liminf_{n\to\infty}\{F_{\epsilon_n}[h_{\epsilon_n};a_n] + F[h_{\epsilon_n};a-a_n]\}\\
&\le \liminf_{n\to\infty}\{F_{\epsilon_n}[h_{\epsilon_n};a_n] + \delta_n\|h_{\epsilon_n}\|_{1,0}\}\\
&\le \liminf_{n\to\infty}\{F_{\epsilon_n}[h_{\epsilon_n};a_n] + \frac{\delta_n}{\sqrt{\epsilon_n}}\sqrt{2\|a\|C\|f\|_{1/2}}\}\\
&\le \liminf_{n\to\infty}\{F_{\epsilon_n}[0;a_n] + \frac{\delta_n}{\sqrt{\epsilon_n}}\sqrt{2\|a\|C\|f\|_{1/2}}\}\\
&= \liminf_{n\to\infty}\{F[0;a_n] + \frac{\delta_n}{\sqrt{\epsilon_n}}\sqrt{2\|a\|C\|f\|_{1/2}}\}\\
&= \liminf_{n\to\infty}\{F[0;a] + F[0;a_n-a] + \frac{\delta_n}{\sqrt{\epsilon_n}}\sqrt{2\|a\|C\|f\|_{1/2}}\}\\
&\le \liminf_{n\to\infty}\{F[0;a] + \delta_n\|\nabla u_0\| + \frac{\delta_n}{\sqrt{\epsilon_n}}\sqrt{2\|a\|C\|f\|_{1/2}}\}\\
&\le \limsup_{n\to\infty}\{F[0;a] + \delta_n C\|f\|_{1/2} + \frac{\delta_n}{\sqrt{\epsilon_n}}\sqrt{2\|a\|C\|f\|_{1/2}}\}\\
&= F[0;a] + \limsup_{n\to\infty}\{\delta_n C\|f\|_{1/2} + \frac{\delta_n}{\sqrt{\epsilon_n}}\sqrt{2\|a\|C\|f\|_{1/2}}\}\\
&= F[0;a].
\end{aligned}
$$

In the above estimate the third inequality uses the minimizing property of $\{h_{\epsilon_n}\}$, while the last equality uses the hypothesis $\delta_n = o(\sqrt{\epsilon_n})$. This proves the identity (21).

We prove next the minimizing property (23) under the additional assumptions that $a \in C(\Omega)$ is bounded from below away from zero. We start by showing that the sequence $\{h_{\epsilon_n}\}$ is bounded in $W_0^{1,1}(\Omega)$. Indeed, from (22)

(25)
$$
\alpha\|h_{\epsilon_n}\|_{1,1} \le \int_\Omega a|\nabla h_{\epsilon_n}|dx \le F[h_{\epsilon_n};a] + F[0;a] \le F[h_{\epsilon_n};a] + C\|a\|\ \|f\|_{1/2}.
$$

It remains to show that the right hand side in (25) can also be bounded uniformly.

$$
\begin{aligned}
F[h_{\epsilon_n};a] \leq & F_{\epsilon_n}[h_{\epsilon_n};a] \\
= & F_{\epsilon_n}[h_{\epsilon_n};a_n] + F[h_{\epsilon_n};a-a_n] \\
\leq & F_{\epsilon_n}[0;a_n] + \delta_n(\|h_{\epsilon_n}\|_{1,0} + C\|f\|_{1/2}) \\
= & F[0;a_n] + \delta_n(\|h_{\epsilon_n}\|_{1,0} + C\|f\|_{1/2}) \\
\leq & \|a_n\|C\|f\|_{1/2} + \delta_n(\|h_{\epsilon_n}\|_{1,0} + C\|f\|_{1/2}) \\
\leq & (\delta_n + \|a\|)C\|f\|_{1/2} + \delta_n(\|h_{\epsilon_n}\|_{1,0} + C\|f\|_{1/2}) \\
= & \|a\|C\|f\|_{1/2} + \frac{\delta_n}{\sqrt{\epsilon_n}}\sqrt{2\|a\|C\|f\|_{1/2}} + 2\delta_n C\|f\|_{1/2} \\
\leq & 3\|a\|C\|f\|_{1/2} + \frac{\delta_n}{\sqrt{\epsilon_n}}\sqrt{2\|a\|C\|f\|_{1/2}}.
\end{aligned}
$$

In the above estimate the second inequality uses the minimization property of $h_{\epsilon_n}$, whereas the last inequality requires $n$ to be sufficiently large. From the hypothesis (20) the last term above is bounded uniformly in $n$.

By the Sobolev imbedding, there is a convergent subsequence (denoted the same) $h_{\epsilon_n} \to \underline{h}$ in $L^q(\Omega)$ for all $1 \leq q \leq d/(d-1)$. Moreover, since $W^{1,1}(\Omega) \subset BV(\Omega)$ we obtain from the lower semicontinuity of the bounded variation norm understood as functionals on continuous maps (it is in here that the continuity assumption on $a$ comes into play) that

$$
\begin{aligned}
\|D(\underline{h}+u_0)\|(a) \leq & \liminf_{n\to\infty} \int_\Omega a(x)|\nabla(h_{\epsilon_n}+u_0)|dx. \\
= & \liminf_{n\to\infty} F[h_{\epsilon_n};a] = F[0;a].
\end{aligned} \tag{26}
$$

Following Corollary 4.2, the inequality (18) extends to all $h \in BV_0(\Omega)$. In particular

$$
F[0;a] = \int_\Omega a|\nabla u_0|dx = \|D(u_0)\|(a) \leq \|D(u_0+\underline{h})\|(a). \tag{27}
$$

□

## 6. Concluding remarks

We showed the stability of the minimum value of the functional $F[\,\cdot\,;a]$ in (4) nearby a specific coefficient $a \in L^2(\Omega)$. More precisely, $a$ is such that the pair $(a,f)$ is admissible. The method of proof constructs a minimizing sequence for $F[\,\cdot\,;a]$ out of minimizers of some regularized functionals. The admissibility is used essentially in establishing the lower bound (24). However, the problem of global stability (near a general coefficient $a$) is of independent interest.

We note that, in general, the minimum value of $F[\,\cdot\,, a]$ may be achieved at $\underline{h} \neq 0$. However, if, a posteriori, $\underline{h} \in W^{1,1}(\Omega)$ and the set of singular points of $\underline{h}+u_0$ is (Lebesgue)-negligible, then the uniqueness result in [8] implies that $\underline{h} = 0$.

The second part of the Theorem 5.1 requires extra regularity on the coefficient, namely $a \in C(\Omega)$. If $\Omega$ is a $C^{1,\alpha}$ domain and $\sigma$ is a $C^{\alpha}(\Omega)$-smooth conductivity, then such a regularity follows automatically from the elliptic regularity up to the boundary by applying a smooth enough boundary data $f \in C^{1,\alpha}(\partial\Omega)$; see, e.g. [3, 6].

**Acknowledgments**
The second author was supported by the NSF grant DMS-0905799.

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