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Optical Tomography in Weakly Anisotropic Scattering Media

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Abstract. This paper considers an inverse boundary value problem for the transport equation in two dimensional domains. The isotropic part of the scattering kernel has no other constrains but sub-criticality. Such scattering kernels are uniquely determined from boundary measurements.

1. Introduction

We start with an unknown strictly-convex two dimensional absorbing and scattering medium Ω. By probing its boundary with (near-infrared) radiation one wants to determine its absorption and scattering properties. This is the method of optical tomography, see [**ARR**] for details. One mathematical model can be formulated as an inverse boundary value problem associated with the transport equation. We distinguish between the incoming and outgoing boundary; if $\widehat{\theta} = (\cos\theta, \sin\theta)$ is a unit direction and $n(x)$ is the outer normal at some boundary point x, we define the incoming, respectively outgoing boundary by $\Gamma_\pm = \{(x,\widehat{\theta}) \in \partial\Omega \times \mathbf{S}^1 : \pm n(x)\cdot\widehat{\theta} > 0\}$. The boundary value problem is

(1.1)
$$\begin{cases} \widehat{\theta}\cdot\nabla_x u(x,\widehat{\theta}) + a(x)u(x,\widehat{\theta}) = \int_0^{2\pi} u(x,\widehat{\theta}')k(x,\theta-\theta')d\theta',\ (x,\widehat{\theta}) \in \Omega\times\mathbf{S}^1, \\ u(x,\widehat{\theta}) = f_-(x,\widehat{\theta}),\ (x,\widehat{\theta}) \in \Gamma_-. \end{cases}$$

Let $\dot{C}^1 = \{k \in C^1(\overline{\Omega}\times[0,2\pi]) : k(x,0) = k(x,2\pi), \partial_\alpha k(x,0) = \partial_\alpha k(x,2\pi)\}$. We work with (sub-critical) scattering kernels in the class

$$U_s = \{k \in \dot{C}^1 : \mathrm{diam}(\Omega)\max\{||k||_\infty, ||\nabla_x k||_\infty\} < 1\}. \tag{1.2}$$

With $k \in U_s$, the boundary value problem (1.1) is well posed and the solution u restricts well to Γ_+. We consider now the Albedo operator which takes f_- (incoming radiation) to the restriction of u on Γ_+ (outgoing radiation).

The inverse problem consists in finding coefficients $a(x)$ and $k(x,\theta)$ from knowledge of the Albedo operator. This problem was first considered by Anikonov and his collaborators [**A, AII, AIII, AIV, APK**]. For three dimensional domains,

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the problem of reconstructing a or k is over-determined and a method of reconstruction was proposed by Choulli and Stefanov in [**CS**]. Stability of their method was proved in the homogeneous case (independence of x) in [**W**].

For two dimensional domains, stability of reconstructing a and k was first obtained by Romanov in [**R**]. When k is a function of two independent directions, its unique determination was proved under assumptions of homogeneity in [**T**]. For an inhomogeneous kernel, which depends on two independent directions, stability was recently established by Stefanov and Uhlmann in [**SU**]. These results are based on the singular decomposition of the Schwartz kernel of the albedo operator. A sub-critical condition is needed to well-define the forward problem, hence the Albedo operator. In addition, the techniques require extra smallness on scattering. It is the latter assumption which distinguish our result. We do not require smallness (other than sub-criticality) on the isotropic part of the scattering $k_0(x) = \int_0^{2\pi} k(x,\theta)d\theta$. However, we do require all the other angular dependent modes to be small. One can view this as a uniqueness result for weakly anisotropic scattering.

Inherent to the optical tomography problem, the two coefficients interlace (1.3). In order to recover a independently of k, we still invoke the result in [**CS**]. Furthermore, a such reconstructed is assumed in $C^2(\overline{\Omega})$.

We use a variational method due to Mukhometov [**M**] applied to Bukhgeim's approach of interpreting the equation in Fourier expansion as a perturbed A-analytic equation [**B**]. The advection operator $\widehat{\theta}\cdot\nabla$ becomes $e^{-i\theta}\overline{\partial}+e^{i\theta}\partial$. Formally expand $u(z,\widehat{\theta}) = \sum_{-\infty}^{\infty} u_n(z)e^{-in\theta}$ and let $\mathbf{u}$ denote the sequence valued function $\mathbf{u}(z) = (u_0(z), u_1(z), u_2(z), ...)$. Since u is real-valued we have $u_n = \overline{u}_{-n}$. Plug into (1.1) the new differential operator and identify coefficients to get the infinite dimensional system

$$\overline{\partial}u_n(z) + \partial u_{n+2}(z) + a(z)u_{n+1}(z) = k_{n+1}u_{n+1}(z),\ n \in \mathbb{Z}. \tag{1.3}$$

Let $L(u_0, u_1, u_2, ...) = (u_1, u_2, ...)$ denote the left translation, and $\overline{\partial} = (\partial_x + i\partial_y)/2$ and $\partial = (\partial_x - i\partial_y)/2$ be the Cauchy-Riemann differential operators. We work on the space l^2 of square sum-able sequences and its Sobolev subspace $l^{2,1} = \{(u_0, u_1, u_2, ...) : \sum_{n=0}^{\infty}(1 + n^2)|u_n|^2 < \infty\}$. Powers of operators are understood as composition.

The system (1.3) for $n \geq 0$ becomes:

$$\overline{\partial}\mathbf{u} + L^2\partial\mathbf{u} + a(z)L\mathbf{u} = LE\mathbf{u}. \tag{1.4}$$

Here $E : C(\overline{\Omega}; l^2) \to C(\overline{\Omega}; l^2)$ is the Fourier multiplier given by

$$E\mathbf{u}(z) = (k_0(z)u_0(z), k_1(z)u_1(z), k_2(z)u_2(z), ...) \tag{1.5}$$

with $k_j(z)$ being the j-th Fourier coefficient of $k(z,\cdot)$. Notice that (1.4) has no $k_0(x)$ involved.

Solutions $\mathbf{u} \in C(\overline{\Omega}; l^{2,1}) \cap C^1(\Omega; l^{2,1})$ of $\overline{\partial}\mathbf{u} + L^2\partial\mathbf{u} = 0$ are called L^2-analytic and they satisfy an analog of Cauchy's integral formula. In particular $\mathbf{u}$ on the boundary determines its values inside. This property was exploited in [**T**] to recover isotropic scattering (k has no angular dependence). For a general theory of A-analytic functions, see Bukhgeim's pioneering work in [**B**].

In [**ABK**] a bounded operator $G : C^1(\overline{\Omega}; l^2) \to C^1(\overline{\Omega}; l^2)$ was constructed such that the equation (1.4) reduces to

$$\overline{\partial}\mathbf{v} + L^2\partial\mathbf{v} = e^{-G}LEe^{G}\mathbf{v}, \tag{1.6}$$

via the transformation $\mathbf{u} = e^G \mathbf{v}$. Moreover G commutes with L and satisfies the Beltrami type equation

$$\overline{\partial} G + L^2 \partial G + aL = 0. \tag{1.7}$$

We briefly recall such a construction. Let $\tau_\pm(x, \widehat{\theta})$ be the travel time in the direction $\pm\widehat{\theta}$ from x to the boundary and let $Da(x, \widehat{\theta})$ be the *divergence beam* transform of a given by

$$Da(x, \widehat{\theta}) = \int_0^{\tau_+(x,\widehat{\theta})} a(x + s\widehat{\theta}) ds.$$

Consider the Fourier expansion of $Da(x, \widehat{\theta})$:

$$Da(z, \widehat{\theta}) = \sum_{-\infty}^{\infty} b_n(z) e^{-in\theta}. \tag{1.8}$$

Since $\tau_\pm$ is smooth (Ω is strictly convex) and $a \in C^2(\overline{\Omega})$, we get $Da \in C^2(\overline{\Omega} \times S^1)$. This regularity ensures that $\sum_{n\in\mathbb{Z}} (1 + n^2)^2 |b_n(z)|^2$, $\sum_{n\in\mathbb{Z}} (1 + n^2) \overline{\partial} |b_n(z)|^2$ and $\sum_{n\in\mathbb{Z}} (1 + n^2) \partial |b_n(z)|^2$ converge uniformly in $\overline{\Omega}$.

The operator G is defined via the odd-rank Fourier coefficients of $Da(x, \cdot)$ by

$$G(z) = \sum_{k=0}^{\infty} b_{-2k-1}(z) L^{2k+1}. \tag{1.9}$$

Since $\widehat{\theta} \cdot \nabla_x Da(z, \widehat{\theta}) = -a(z)$ and using the (non trivial) fact that $\overline{\partial} b_1(z) = 0$ (see [**TII**]), one can check (1.7) and the mapping properties of G and e^{-G}. For example, Lemma 2.3 (with $\alpha = 1$) renders $G : C(\overline{\Omega}; l^2) \to C(\overline{\Omega}; l^2)$ bounded. For additional details refer to [**TII**].

For each $a \in C^2(\overline{\Omega})$ we define a norm on a via the operator norm of G by

$$|||a||| = \sup_{\overline{\Omega}} \{ ||G(z)\mathbf{v}|| : \mathbf{v} \in l^2, \ ||\mathbf{v}|| = 1 \}, \tag{1.10}$$

where $|| \cdot ||$ is the l^2-norm.

The result of this article is the following.

Theorem 1.1. *Define the class*

$$U^\alpha_{\sigma,\epsilon} = \{ (a, k) \in C^2(\overline{\Omega}) \times U_s : |||a||| \le \sigma, (1 + j)^\alpha ||k_j||_\infty \le \epsilon, j \ge 1 \}.$$

For any $\sigma > 0$ and $\alpha > 1/2$ there exists an $\epsilon > 0$ such that a pair $(a, k) \in U^\alpha_{\sigma,\epsilon}$ is uniquely determined by the albedo operator in its class.

2. Gradient estimates for solutions of Beltrami equations

In the following three lemmas we denote by $< \cdot, \cdot >$ the l^2-inner product, and by $|| \cdot ||$ its corresponding norm.

Lemma 2.1. *Let Ω be a planar domain with piecewise-C^1 boundary. For $\mathbf{v} \in C(\overline{\Omega}; l^2) \cap C^1(\Omega; l^2)$ we have*

$$2 \int_\Omega ||\partial \mathbf{v}||^2 dx = 2 \int_\Omega ||\overline{\partial} \mathbf{v}||^2 dx + i \int_{\partial\Omega} < \mathbf{v}, \partial_s \mathbf{v} > ds, \tag{2.1}$$

where ∂_s is the tangential derivative at the boundary.

PROOF. For two functions f, g rewrite Green's formula

$$2\int_\Omega \overline{\partial} f \overline{g} dx = \int_{\partial\Omega} \nu f \overline{g} ds - 2\int_\Omega f \overline{\partial}\overline{g} dx,$$
$$2\int_\Omega \partial f \overline{g} dx = \int_{\partial\Omega} \overline{\nu} f \overline{g} ds - 2\int_\Omega f \partial\overline{g} dx.$$

In the first equation let f range over each component of $\mathbf{v}$ and g range over the corresponding component of $\overline{\partial}\mathbf{v}$. In the second equation, let f range over the components of $\mathbf{v}$ and g range over the corresponding $\partial\mathbf{v}$. Sum each of the equations over the components, then subtract them to get

$$2\int_\Omega ||\partial\mathbf{v}||^2 dx = 2\int_\Omega ||\overline{\partial}\mathbf{v}||^2 dx + \int_{\partial\Omega} < \mathbf{v}, (\nu\partial - \overline{\nu}\overline{\partial})\mathbf{v} > ds.$$

Since $(\nu\partial - \overline{\nu}\overline{\partial}) = -i\partial_s$ the lemma is proved. □

Let R be the adjoint of L, defined by $R(v_0, v_1, ...) = (0, v_0, v_1, ...)$ and $P_n = R^n L^n$ be the projection onto the n-th tail.

LEMMA 2.2 (Bukhgeim's Identity). *Let $\mathbf{v} \in C^1(\overline{\Omega}; l^{2,1})$ be a solution of*

$$\overline{\partial}\mathbf{v} + L^2\partial\mathbf{v} = B\mathbf{v}, \tag{2.2}$$

where $B : l^2 \to l^2$ is a bounded operator. The following identity holds

$$2\int_\Omega ||\partial\mathbf{v}||^2 dx = -2\mathbb{R}e\int_\Omega < \partial\mathbf{v}, \sum_{j=1}^\infty R^{2j} B L^{2j-2}\mathbf{v} > dx + \tag{2.3}$$
$$2\sum_{j=0}^\infty \int_\Omega ||BL^{2j}\mathbf{v}||^2 dx + i\sum_{j=0}^\infty \int_{\partial\Omega} < \mathbf{v}, P_{2j}\partial_s\mathbf{v} > ds.$$

PROOF. We first show that all the sums in the right hand side are finite. Indeed for every $z \in \overline{\Omega}$ the following inequality holds:

$$\sum_{j=0}^\infty \left(\sum_{k=0}^\infty |v_{j+k}(z)| \right)^2 \leq 4\sum_{j=0}^\infty (1+j^2)|v_j(z)|^2. \tag{2.4}$$

We prove first its continuous version $\int_1^\infty \left(\int_s^\infty f(t)dt\right)^2 ds \leq 4\int_1^\infty (1+s^2) f^2(s) ds$, then set $f(t) = |c_{[t]}|$, for$[t]$ the largest integer smaller then t:

$$\left\{ \int_1^\infty \left(\int_s^\infty f(t)dt \right)^2 ds \right\}^{\frac{1}{2}} = \left\{ \int_1^\infty \left(\int_1^\infty s f(s\zeta) d\zeta \right)^2 ds \right\}^{\frac{1}{2}} \leq$$
$$\int_1^\infty \left\{ \int_1^\infty s^2 f^2(s\zeta) ds \right\}^{\frac{1}{2}} d\zeta = \int_1^\infty \left\{ \int_\zeta^\infty \frac{t^2}{\zeta^3} f^2(t) dt \right\}^{\frac{1}{2}} d\zeta \leq$$
$$\int_1^\infty \zeta^{-\frac{3}{2}} d\zeta \left\{ \int_1^\infty t^2 f^2(t) dt \right\}^{\frac{1}{2}} \leq 2\left\{ \int_1^\infty (1+t^2) f^2(t) dt \right\}^{\frac{1}{2}}.$$

From Green's identity (2.1) and equation (2.2) we get

$$2\int_\Omega ||\partial\mathbf{v}||^2 dx = 2\int_\Omega ||L^2\partial\mathbf{v} - B\mathbf{v}||^2 dx + i\int_{\partial\Omega} < \mathbf{v}, \partial_s\mathbf{v} > ds = 2\int_\Omega ||L^2\partial\mathbf{v}||^2 dx -$$
$$2\int_\Omega \mathbb{R}e < \partial\mathbf{v}, R^2 B\mathbf{v} > dx + 2\int_\Omega ||B\mathbf{v}||^2 dx + i\int_{\partial\Omega} < \mathbf{v}, \partial_s\mathbf{v} > ds.$$

For $j = 1, ..., n$ successively replace in the identity above $\mathbf{v}$ by $L^{2j}\mathbf{v}$ and add them together:

$$2\int_\Omega ||\partial\mathbf{v}||^2 dx = 2\int_\Omega ||L^{2n}\partial\mathbf{v}||^2 dx - 2\mathbb{R}e\int_\Omega < \partial\mathbf{v}, \sum_{j=1}^n R^{2j}BL^{2j-2}\mathbf{v} > dx +$$
$$2\sum_{j=0}^n \int_\Omega ||BL^{2j}\mathbf{v}||^2 dx + i\sum_{j=0}^n \int_{\partial\Omega} < \mathbf{v}, P_{2j}\partial_s\mathbf{v} > ds.$$

Now use $\lim_{n\to\infty}\int_\Omega ||L^{2n}\partial\mathbf{v}||^2 dx = 0$ to conclude the lemma. □

The following lemma provides the key inequality for our estimate. Recall the notation $EL^n\mathbf{v} = EL^n(v_0, v_1, ...) = (e_0 v_n, e_1 v_{n+1,...})$.

Lemma 2.3. *If* $|e_j| \le C(1+j)^{-\alpha}$ *for some* $\alpha > 1/2$ *and* $j = 1, 2, ...,$ *then*

$$\sum_{n=0}^\infty ||EL^n\mathbf{v}||^2 \le \frac{2C}{2\alpha-1}||\mathbf{v}||^2. \tag{2.5}$$

Proof. Equivalently, we need to show that

$$\sum_{j=0}^\infty\sum_{n=0}^\infty \frac{1}{(1+j)^{2\alpha}}|v_{j+n}|^2 \le \frac{2}{2\alpha-1}\sum_{j=0}^\infty |v_j|^2.$$

We show first its continuous version:

$$\int_0^\infty dt\left\{\int_t^\infty \frac{f^2(s)}{(1+t)^{2\alpha}}ds\right\} = \int_0^\infty ds\left\{f^2(s)\int_0^s dt\frac{1}{(1+t)^{2\alpha}}\right\} =$$
$$= \frac{1}{2\alpha-1}\int_0^\infty f^2(s)\left(1 - \frac{1}{(1+s)^{2\alpha-1}}\right)ds \le \frac{1}{2\alpha-1}\int_0^\infty f^2(s)ds.$$

Now choose $f(s) = |v_{[s]}|$, where $[s] \le s < [s]+1$ and notice that

$$\int_0^\infty dt\left\{\int_t^\infty \frac{f^2(s)}{(1+t)^{2\alpha}}ds\right\} = \sum_{j=0}^\infty \int_j^{j+1} dt\left\{\int_t^\infty \frac{f^2(s)}{(1+t)^{2\alpha}}ds\right\} \ge$$
$$\ge \sum_{j=0}^\infty \int_j^{j+1}\frac{dt}{(2+j)^{2\alpha}}\int_j^\infty f^2(s)ds \ge \frac{1}{2}\sum_{j=0}^\infty \frac{1}{(1+j)^{2\alpha}}\sum_{n=0}^\infty |v_{n+j}|^2.$$

□

Using this lemma, we are ready to formulate the gradient estimate

Theorem 2.4. *Let* $E : C(\overline{\Omega}; l^2) \to C(\overline{\Omega}; l^2)$ *be as in* (1.5) *and such that for some* $\alpha > 1/2$ *and each* $j \ge 1$*:*

$$\sup_{\overline{\Omega}} |k_j(z)| \le C(1+j)^{-\alpha}. \tag{2.6}$$

Let $G : C^1(\Omega; l^{2,1}) \to C^1(\Omega; l^{2,1})$ *be given by* (1.9) *and* $\sigma = \sup_{\overline{\Omega}}\sup\{||G\mathbf{v}|| : \mathbf{v} \in l^2, ||\mathbf{v}|| = 1\}$. *If* $\mathbf{v} \in C^1(\overline{\Omega}; l^{2,1})$ *satisfies* $\overline{\partial}\mathbf{v} + L^2\partial\mathbf{v} = e^{-G}LEe^G\mathbf{v}$, *then*

$$\int_\Omega ||\partial\mathbf{v}||^2 dx \le \frac{12Ce^{2\sigma}}{2\alpha-1}\int_\Omega ||\mathbf{v}||^2 dx + 4\int_{\partial\Omega} ||\partial_s\mathbf{v}||^2_{2,1}, \tag{2.7}$$

where $||\cdot||_{2,1}$ *is the* $l^{2,1}$*-norm and* ∂_s *is the tangential derivative at the boundary.*

Before proceeding to the proof remark that there are no assumptions on the size of $k_0(z)$. Note also that $\sigma = |||a|||$ as defined in (1.10).

PROOF. We estimate each term in the right hand side of Bukhgeim's identity (2.3) for $B = e^{-G}LEe^{G}$:

$$2\int_\Omega |\left\langle \partial\mathbf{v}, \sum_{j=1}^\infty R^{2j}BL^{2j-2}\mathbf{v}\right\rangle|dx \leq \int_\Omega ||\partial\mathbf{v}||^2dx + 4\sum_{j=1}^\infty \int_\Omega ||BL^{2j-2}\mathbf{v}||^2dx,$$

$$\sum_{j=0}^\infty \int_{\partial\Omega} < \mathbf{v}, P_{2j}\partial_s\mathbf{v} > ds \leq 4\int_{\partial\Omega} ||\partial_s\mathbf{v}||^2_{2,1}ds.$$

For the first inequality above, we used that the operator norm of R and its powers equal one, whereas for the second inequality we appeal to (2.4).

In Bukhgeim's identity take absolute values for the terms in the right hand side and use the two estimates above and the definition of B. We get

$$\begin{aligned}\int_\Omega ||\partial\mathbf{v}||^2dx \leq & 6\sum_{j=0}^\infty \int_\Omega ||e^{-G}LEL^{2j}e^{G}\mathbf{v}||^2dx + 4\int_{\partial\Omega} ||\partial_s\mathbf{v}||^2_{2,1}ds \\ \leq & 6e^\sigma \sum_{j=0}^\infty \int_\Omega ||LEL^{2j}e^{G}\mathbf{v}||^2dx + 4\int_{\partial\Omega} ||\partial_s\mathbf{v}||^2_{2,1}ds \\ \leq & 6e^\sigma \frac{2C}{2\alpha-1}\int_\Omega ||e^{G}\mathbf{v}||^2dx + 4\int_{\partial\Omega} ||\partial_s\mathbf{v}||^2_{2,1}ds \\ \leq & \frac{12Ce^{2\sigma}}{2\alpha-1}\int_\Omega ||\mathbf{v}||^2dx + 4\int_{\partial\Omega} ||\partial_s\mathbf{v}||^2_{2,1}ds.\end{aligned}$$

□

The following corollary is an immediate application to Poicaré 's inequality.

COROLLARY 2.5. Under the assumptions of the theorem 2.4, if C in (2.6) is sufficiently small, then

$$\int_\Omega ||\partial\mathbf{v}||^2dx \leq \tilde{C}\int_{\partial\Omega}(||\mathbf{v}||^2 + ||\partial_s\mathbf{v}||^2_{2,1})ds, \tag{2.8}$$

for some $\tilde{C}$ which depends on α, C and σ as well as on the geometry of Ω.

PROOF. From Poincaré' s inequality applied to each component of $\mathbf{v}$, there is a constant β depending only on Ω, such that for every $j \geq 0$:

$$\int_\Omega |v_j|^2dx \leq \beta\left(\int_\Omega |\partial v_j|^2dx + \int_{\partial\Omega}|v_j|^2ds\right).$$

Using these into the energy estimate (2.7) we get

$$\int_\Omega ||\partial\mathbf{v}||^2dx \leq \frac{12\beta Ce^{2\sigma}}{2\alpha-1}\int_{\partial\Omega}(||\partial\mathbf{v}||^2 + ||\mathbf{v}||^2)ds + 4\int_{\partial\Omega}||\partial_s\mathbf{v}||^2_{2,1}ds.$$

Now choose C so that

$$C < \frac{e^{-2\sigma}(2\alpha-1)}{12\beta}, \tag{2.9}$$

and define

$$\tilde{C} = 4\left(1 - \frac{12\beta C e^{2\sigma}}{2\alpha - 1}\right)^{-1}.$$

□

3. Unique determination of the scattering kernel

The result of this section comes as a consequence of the gradient estimate (2.8). The only concern left is whether (and under which assumptions) we have solutions of the transport equation such that the corresponding $\mathbf{v} \in C^1(\overline{\Omega}, l^{2,1})$. Since G and $e^{\pm G}$ maps $C^1(\overline{\Omega}, l^{2,1})$ to itself (see [**TII**] for details), it suffices to exhibit solutions $u(x, \widehat{\theta})$ of (1.1) in $C^1(\overline{\Omega} \times \mathbf{S}^1)$. The relation between smoothness of a function and decay properties of its Fourier coefficients conclude the result.

We consider $C^1(\overline{\Omega} \times \mathbf{S}^1)$ endowed with the norm

$$||u||_{1,\infty} = (1 + ||\nabla a||_\infty)||u||_\infty + ||\nabla_x u||_\infty. \tag{3.1}$$

For every $f_- \in C^1(\Gamma_-)$ define $Jf_- \in C^1(\overline{\Omega} \times \mathbf{S}^1)$ by

$$Jf_-(x, \widehat{\theta}) = f_-(x - \tau_-(x, \widehat{\theta})\widehat{\theta}, \widehat{\theta}) e^{-\int_0^{\tau_-(x,\widehat{\theta})} a(x - s\widehat{\theta})ds},$$

so that it solves the homogeneous equation $[\widehat{\theta} \cdot \nabla_x + a]Jf_- = 0$.

Consider the equation (1) evaluated at $(x - t\widehat{\theta}, \widehat{\theta})$ and multiplied by the integrand factor

$$\omega(t, x, \widehat{\theta}) = e^{-\int_0^t a(x - s\widehat{\theta})ds}.$$

Integrating in t from 0 to $+\infty$ and taking into account the boundary condition we get the integral equation

$$[I - K]u = Jf_-, \tag{3.2}$$

where

$$Ku(x, \widehat{\theta}) = \int_0^\infty \omega(t, x, \widehat{\theta}) \int_0^{2\pi} k(x - t\widehat{\theta}, \theta - \theta')u(x, \widehat{\theta}')d\theta' dt.$$

The following lemma provides a smooth solution to (1.1).

Lemma 3.1. *Let $M = \max\{||k||_\infty, ||\nabla_x k||_\infty\}$. If $M\, diam(\Omega) < 1$ then $K : C^1(\overline{\Omega} \times \mathbf{S}^1) \to C^1(\overline{\Omega} \times \mathbf{S}^1)$ is a contraction with respect to $|| \cdot ||_{1,\infty}$ from* (3.1).

Proof. Since $a \geq 0$ we can bound the exponential by one and then

$$||Ku||_\infty \leq \mathrm{diam}(\Omega)||k||_\infty ||u||_\infty. \tag{3.3}$$

On the other hand when taking the derivative in x_i for $i = 1, 2$, we have

$$\begin{aligned}\frac{\partial Ku}{\partial x_i}(x, \widehat{\theta}) = \int_0^\infty \omega(t, x, \widehat{\theta}) \int_0^t \frac{\partial a}{\partial x_i}(x - s\widehat{\theta})ds \int_0^{2\pi} k(x - t\widehat{\theta}, \theta - \theta')u(x, \widehat{\theta}')d\theta' dt \\ + \int_0^\infty \omega(t, x, \widehat{\theta}) \int_0^{2\pi} \frac{\partial k}{\partial x_i}(x - t\widehat{\theta}, \theta - \theta')u(x, \widehat{\theta}')d\theta' dt \\ + \int_0^\infty \omega(t, x, \widehat{\theta}) \int_0^{2\pi} k(x - t\widehat{\theta}, \theta - \theta')\frac{\partial u}{\partial x_i}(x, \widehat{\theta}')d\theta' dt.\end{aligned}$$

Taking the supremum over $(x,\widehat{\theta}) \in \overline{\Omega}\times \mathbf{S}^1$ implies

$$||\partial_{x_i}Ku||_\infty \leq \mathrm{diam}(\Omega)\left(||\nabla a||_\infty ||k||_\infty + ||\nabla_x k||_\infty\right)||u||_\infty + \mathrm{diam}(\Omega)||k||_\infty||\nabla_x u||_\infty \tag{3.4}$$

$$\leq M\mathrm{diam}(\Omega)\left((1+||\nabla a||_\infty)||u||_\infty + ||\nabla_x u||_\infty\right) \tag{3.5}$$

$$= M\mathrm{diam}(\Omega)||u||_{1,\infty}. \tag{3.6}$$

From (3.3) and (3.4) we have $||Ku||_{1,\infty} \leq M\mathrm{diam}(\Omega)||u||_{1,\infty}$. □

We have all the ingredients needed to prove the uniqueness result.

Proof of the Theorem 1.1. Choulli and Stefanov in [**CS**] showed that the main singularity in the Schwartz kernel of the Albedo operator determines a independently of k.

For given $\sigma > 0$ and $\alpha > 1/2$, choose an ϵ as in (2.9), i.e.

$$\epsilon < \frac{e^{-2\sigma}(2\alpha-1)}{12\beta}.$$

Let u and $\tilde{u}$ be two solutions of (1.1) given by the lemma 3.1 corresponding to two pairs (a,k) respectively $(a,\tilde{k})$ in $\mathcal{U}_{\sigma,\epsilon,\alpha}$. Let $\mathbf{v} = e^{-G}\mathbf{u}$ and $\tilde{\mathbf{v}} = e^{-G}\tilde{\mathbf{u}}$ where $\mathbf{u} = (u_0, u_1, ...)$ and $\tilde{\mathbf{u}} = (\tilde{u}_0, \tilde{u}_1, ...)$ collect the non-negative Fourier coefficients. Then their difference $\mathbf{v}-\tilde{\mathbf{v}}$ satisfies the Beltrami equation $\overline{\partial}\mathbf{v}+\mathcal{L}^2\partial\mathbf{v} = e^{-G}\mathcal{L}Ee^G\mathbf{v}$. Since $\mathbf{v} = \tilde{\mathbf{v}}$ on the boundary $\partial\Omega$, by (2.8) we get $\partial\mathbf{v} = \partial\tilde{\mathbf{v}}$ in Ω. Since they also match on the boundary $\mathbf{v} = \tilde{\mathbf{v}}$. Provided $u_j \neq 0$ the system (1.3) shows how to recover each $k_j(x)$ from $\mathbf{u}$ and a. This condition can be satisfied if applying an incoming flux of non-vanishing $j-th$ Fourier coefficient. There is a simple criterion consequence of Stokes' theorem. If $u_j = 0$ in Ω, then (1.3) implies that $\overline{\partial}u_{j-1} = -\partial u_{j+1}$. Therefore the 1-form $u_{j-1}dz - u_{j+1}d\overline{z}$ is exact and

$$\int_{\partial\Omega} u_{j-1}dz - u_{j+1}d\overline{z} = 0.$$

□

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