Hadwiger Number and Chromatic Number for Near Regular Degree Sequences

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June 30, 2009

Abstract

We consider a problem related to Hadwiger's Conjecture. Let $D = (d_1, d_2, \dots, d_n)$ be a graphic sequence with $0 \leq d_1 \leq d_2 \leq \cdots \leq d_n \leq n-1$. Any simple graph G with D its degree sequence is called a realization of D. Let $R[D]$ denote the set of all realizations of D. Define $h(D) = \max\{h(G) : G \in R[D]\}$ and $\chi(D) = \max\{\chi(G) : G \in R[D]\}$, where $h(G)$ and $\chi(G)$ are Hadwiger number and chromatic number of a graph G, respectively. Hadwiger's Conjecture implies that $h(D) \geq \chi(D)$. In this paper, we establish the above inequality for near regular degree sequences.

Key Words : Hadwiger Number, Chromatic Number, Graphic Degree Sequence
Mathematics Subject Classification 2000 : 05C83, 05C15, 05C07

*Partially supported by NSA grant H98230-09-1-0055

1 Introduction

All graphs in this paper are finite and have no loops or multiple edges. A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. An H *minor* is a minor isomorphic to H. If H is a complete graph, we also say that G contains a *clique minor* of size $|H|$. For a graph G, the *Hadwiger number* $h(G)$ of G is the maximum integer k such that G contains a clique minor of size k. As usual we denote by $\chi(G)$ the chromatic number of G and by $\chi'(G)$ the edge chromatic number of G.

Our research is motivated by Hadwiger's Conjecture from 1943 which suggests a far-reaching generalization of the Four Color Theorem [1, 2, 7] and is considered by many as one of the deepest open problems in graph theory. Hadwiger's Conjecture states the following.

Conjecture 1.1 *For every integer $k \geq 1$, every k-chromatic graph has a K_k minor.*

Conjecture 1.1 is trivially true for $k \leq 3$, and reasonably easy for $k = 4$, as shown by Dirac [5] and Hadwiger himself [6]. However, for $k \geq 5$, Conjecture 1.1 implies the Four Color Theorem. In 1937, Wagner [10] proved that the case $k = 5$ of Conjecture 1.1 is, in fact, equivalent to the Four Color Theorem. In 1993, Robertson, Seymour and Thomas [8] proved that a minimal counterexample to the case $k = 6$ is a graph G which has a vertex v such that $G - v$ is planar. By the Four Color Theorem, this implies Conjecture 1.1 for $k = 6$. Hence the cases $k = 5, 6$ are each equivalent to the Four Color Theorem [1, 2, 7]. Conjecture 1.1 is open for $k \geq 7$. Note that Conjecture 1.1 also states that for every graph G, $h(G) \geq \chi(G)$.

In this paper, we consider a weaker version of Hadwiger's Conjecture. The goal is to establish more evidence for Conjecture 1.1. Let $D = (d_1, d_2, \ldots, d_n)$ be an integer sequence with $0 \leq d_1 \leq d_2 \leq \cdots \leq d_n \leq n - 1$. We say that D is *graphic* if there is a graph G with $V(G) = \{u_1, u_2, \cdots, u_n\}$ such that $d_G(u_i) = d_i$. In those circumstances, we say that G is a *realization* of D. If a sequence D consists of the terms $d_1, \ldots, d_t$ with multiplicities $m_1, \ldots, m_t$, we may write $D = (d_1^{m_1}, \ldots, d_t^{m_t})$. For a graphic degree sequence D, let $R[D]$ denote the set of all realizations of D. Define $h(D) = max\{h(G) : G \in R[D]\}$ and $\chi(D) = max\{\chi(G) : G \in R[D]\}$. Hadwiger's Conjecture implies the following conjecture.

Hadwiger's Conjecture for Degree Sequences: For any graphic degree sequence D, $h(D) \geq \chi(D)$.

For a graphic sequence $D = (d_1, d_2, \ldots, d_n)$, we say that D is near regular if $D = ((k-1)^p, k^{n-p})$ for some integers $k \geq 1$ and p satisfying $0 \leq p \leq n - 1$. The purpose of this paper is to prove that Hadwiger's Conjecture for Degree Sequences is true for near regular degree sequences. Our proof technique is to construct a graph $G \in R[D]$ with a clique minor of desired size.

We need to introduce more notation. Let G be a graph. The *complement* of G is denoted by $\overline{G}$. If $X \subseteq V(G)$, we denote the subgraph of G induced on X by $G[X]$. We use $G \backslash X$ denote the subgraph of G induced on $V(G) \backslash X$. If $A, B \subseteq V(G)$ are disjoint, we say that A is *complete* to B if every vertex in A is adjacent to every vertex in B, and A is *anticomplete* to B if no vertex in A is adjacent to a vertex in B. If $A = \{x\}$, we simply say x is complete to B or x is anticomplete to B. We use $G[A, B]$ to denote the bipartite graph obtained from $G[A \cup B]$ by deleting all edges with both ends in A or in B. If $F \subseteq E(G)$ and $M \subseteq E(\overline{G})$, then $G - F$ (resp. $G + M$) denotes the graph obtained from G by deleting the edges in F from G (reps. adding the edges in M to G). For two disjoint graphs G and H, $G + H$ is the *join* of G and H with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$. As usual we denote by K_n and $K_{n,m}$, respectively, the complete graph on n vertices and the complete bipartite graph such that one partite set has n vertices and the other partite set has m vertices.

The following well-known results [3, 9] will be used later in the proof of the main results. We list them below. A proof of Hall's Theorem and other notation not introduced here can be found in [4].

Theorem 1.2 *Let G be a connected graph with maximum degree Δ. Suppose G is neither a complete graph nor an odd cycle. Then $\chi(G) \leq \Delta$.*

Theorem 1.3 *If G is a r-regular bipartite graph with $r \geq 1$, then G has r pairwise disjoint perfect matchings.*

Theorem 1.4 *If G is simple graph with maximum degree r, then $\chi'(G) = r + 1$ or r.*

2 Near Regular Degree Sequences

In this section, we prove that Hadwiger's Conjecture for Degree Sequences is true for near regular degree sequences. We first prove some preliminary results.

Lemma 2.1 *For integers $r \geq 2$ and $n \geq r + 1$, if nr is even, then there exists an r-regular graph G of order n such that G has at least two pairwise disjoint perfect matchings if n is even and one near perfect matching if n is odd.*

Proof. If $r = n - 1$, then $G = K_n$ has the desired property. So we may assume that $r \leq n - 2$. Assume that n is even. If $r \leq \frac{n}{2}$, let G be an r-regular bipartite graph with each partite of size $\frac{n}{2}$. If $r \geq \frac{n}{2} + 1$, let G be a graph with $V(G)$ partitioned into A and B such that $G[A] = G[B] = K_{\frac{n}{2}}$ and $G[A, B]$ is s-regular, where $s = r - (\frac{n}{2} - 1) \geq 2$. In either case, by Theorem 1.3, G has at least two pairwise disjoint perfect matchings. If n is odd, then r and $n - 1$ must be even. Let H be the r-regular graph of order $n - 1$, as constructed above, with two disjoint perfect matchings, say M_1, M_2. Let $F \subseteq M_1$ with $|F| = \frac{r}{2}$. We may assume that $F = \{x_1x_2, x_3x_4, \ldots, x_{r-1}x_r\}$. Let G be obtained from $H - F$ by joining a new vertex w to x_i, $i = 1, 2, \ldots, r$. Clearly G is r-regular and M_2 is a near perfect matching of G.

This completes the proof of Lemma 2.1. ∎

Lemma 2.2 *Let $D = ((k-1)^p, k^{n-p})$ be the degree sequence of a near k-regular graph on n vertices, where $0 \leq p \leq n-1$. If $3n \geq 4k+4$, then $h(D) \geq k+1$ if $p = 0$ and $h(D) \geq k$ if $p > 0$.*

Proof. The statement is trivially true if $k \leq 2$. So we may assume that $k \geq 3$. We consider the following two cases.

Case 1. n is even.

Since $\sigma(D) = nk - p$ is even, p must be an even integer. Suppose that $n \geq 2k+2$. If $n - p \geq k+1$, by Lemma 2.1, let H be a k-regular graph of order $n-k-1$ and let F be a matching of size $\frac{p}{2}$ in H. Then $K_{k+1} \cup G \in R[D]$, where $G = H - F$. Thus $h(D) \geq h(K_{k+1} \cup G) \geq k+1$. If $n - p \leq k$, by Lemma 2.1, let H be a $(k-1)$-regular graph of order $n-k$ and let F be a matching of size $\frac{n-p}{2}$ in $\overline{H}$. Then $K_k \cup G \in R[D]$, where $G = H + F$. Hence $h(D) \geq h(K_k \cup G) \geq k$. So we may assume that $n \leq 2k$. For integers r, s, t satisfying $n = 2s + 2t$ and $r = s + 2t - 1$, define $G(r, s, t)$ to be an r-regular graph of order n with its vertex set partitioned into A, B, C, D such that $G(r, s, t)[A] = G(r, s, t)[B] = K_s$, $G(r, s, t)[C] = G(r, s, t)[D] = K_t$, $G(r, s, t)[A, B]$ is a t-regular bipartite graph. Moreover, C is complete to $A \cup D$, D is complete to B, A is anticomplete to D, C is anticomplete to B.

If $p \leq 2(n-k-1)$, let $s = n-k-1$ and $t = k+1-\frac{n}{2}$. Then $\frac{k}{3} < s \leq k-1$, $1 \leq t < \frac{k}{3}$, and $s - \frac{p}{2} \geq 0$. Observe that $2s + 2t = n$, $s + 2t = k+1$ and $t \leq s$ because $3n \geq 4k+4$. Let $G = G(k, s, t)$. Then G is a k-regular graph of order n. If $t = 1$, then $n = 2k$ and so $s + t = k$. Let M be a perfect matching of $G[A, B]$ and $F \subseteq M$ with $|F| = \frac{p}{2}$. Let $H = G - F$. Then $H \in R[D]$. By contracting each of the edges of $M - F$ into single vertices, we have $h(D) \geq h(H) \geq |A| + |C| + |D| = s + 2t = k+1$ if $p = 0$, and $h(D) \geq h(H) \geq |A| + |C| = s + t = k$ if $p > 0$. So we may assume that $t \geq 2$. By Theorem 1.3, $G[A, B]$ contains two disjoint perfect matchings M_1 and M_2. Let $F \subset M_1$ with $|F| = \frac{p}{2}$ and let $H = G - F$. Then $H \in R[D]$. By contracting each of the edges of M_2 into single vertices, we see that $h(D) \geq h(H) \geq s + 2t = k+1$.

So we may assume that $p \geq 2(n-k)$. Then $n - p \leq n - 2(n-k) = 2k - n < 2(n-k)$ because $3n \geq 4k+4$. Note that $n-p$ is even. Let $s = n-k$ and $t = k - \frac{n}{2}$. Then $\frac{k}{3} < s \leq k$, $0 \leq t < \frac{k}{3}$. Observe that $2s + 2t = n$, $s + 2t = k$ and $t < s$ because $3n \geq 4k+4$. Let $G = G(k-1, s, t)$. Then G is a $(k-1)$-regular graph of order n. Since $t < s$, by Theorem 1.3, $G[A, B]$ contains an anti-matching F of size $\frac{n-p}{2}$. Let $H = G + F$, then $H \in R[D]$. If $t = 0$, then $n = 2k$ and so $h(D) \geq h(H) \geq |A| = s = k$. If $t \geq 1$, let M be a perfect matching of

$G[A,B]$. Clearly, M is also a matching of H. By contracting each of the edges of M into single vertices, we see that $h(D) \geq h(H) \geq s+2t = k$.

Case 2. n is odd.

In this case, since $\sigma(D) = nk - p$ is even, we have

(a) p and k are either both even or both odd.

By (a), both $p-k$ and $n-p-k-1$ are even. Suppose that $n \geq 2k+1$. If $n-p \leq k$, then $p-k \leq 2(n-k)$. By Lemma 2.1, let H be a k-regular graph of order $n-k$ and let F be a matching of size $\frac{p-k}{2}$ in H. Then $K_k \cup G \in R[D]$, where $G = H - F$. Hence $h(D) \geq h(K_k \cup G) \geq k$. We may assume that $n-p \geq k+1$. If $n \geq 2k+3$, by Lemma 2.1, let H be a $(k-1)$-regular graph of order $n-k-1 \geq k+2$ and let F be a matching of size $\frac{n-p-k-1}{2}$ of $\overline{H}$. Then $K_{k+1} \cup G \in R[D]$, where $G = H+F$. Thus $h(D) \geq h(K_{k+1} \cup G) \geq k+1$. We may assume that $n = 2k+1$. Then $p \leq n-k-1 = k$. Let $F = \{x_1x_2, \ldots, x_{2q-1}x_{2q}\}$ be a matching of size q of K_{k+1}, where $q = \frac{n-p-k-1}{2} = \frac{k-p}{2}$. Let $y_1, y_2, \ldots, y_{2q}$ be $2q$ distinct vertices in K_k. Let G be the graph obtained from $K_{k+1} - F$ and K_k by joining y_i to x_i, $i = 1, 2, \ldots, 2q$. Then $G \in R[D]$. By contracting $G[\{y_{2i-1}, y_{2i}\}]$ onto x_{2i-1}, $i = 1, 2, \ldots, q$, we see that $h(D) \geq h(G) \geq k+1$.

So we may assume that $n \leq 2k-1$. For integers r, s, t satisfying $n = 2s+2t+1$ and $r = s+2t$, define $I(r,s,t)$ to be a graph of order n with its vertex set partitioned into A, B, C, D such that $I(r,s,t)[A] = K_s$, $I(r,s,t)[B] = K_s - E$, $I(r,s,t)[C] = K_t$, $I(r,s,t)[D] = K_{t+1}$, $I(r,s,t)[A,B]$ is a $(t+1)$-regular bipartite graph, where E is matching of size $[\frac{s}{2}]$ of K_s. Moreover, C is complete to $A \cup D$, D is complete to B, A is anticomplete to D, C is anticomplete to B. By (a), we consider the following two subcases.

Case 2.1 p and k are even.

If $p \leq 2(n-k-1)$, let $s = n-k-1$ and $t = k+1-\frac{n+1}{2}$. Then $\frac{k}{3} < s \leq k-2$, $1 \leq t < \frac{k}{3}$. Observe that s is even, $s+2t = k$ and $t \leq s$. Let $G = I(k,s,t)$. Then G is a k-regular graph of order n. Since $t \geq 1$, by Theorem 1.3, let M_1 and M_2 be two perfect matchings of $G[A,B]$. Let $F \subseteq M_1$ with $|F| = \frac{p}{2}$. Let $H = G - F$. Then $H \in R[D]$. By contracting each of the edges of M_2 into single vertices, we see that $h(H) \geq s+2t+1 = k+1$. Hence $h(D) \geq k+1$.

So we may assume that $p \geq 2(n-k)$. Then $n-p \leq n-2(n-k) = 2k-n < 2(n-k)$ and $n-p$ is odd. Let $s = n-k$ and $t = k - \frac{n+1}{2}$. Then $\frac{k}{3} \leq s \leq k-1$, $0 \leq t < \frac{k}{3}$. Observe that s is odd, $s+2t = k-1$ and $t < s$ because $3n \geq 4k+4$. Let $G = I(k-1,s,t)$. Then $DS(G) = (k^1, (k-1)^{n-1})$. Let w be the vertex of degree $s-1$ in $G[B]$. Since $n-p < 2s$, by Theorem 1.3, $G[A, B\backslash w]$ contains an anti-matching, say F, of size $\frac{n-p-1}{2}$. Let $H = G+F$. Then $H \in R[D]$. Let M be a perfect matching of $G[A,B]$. By contracting each of the edges of M into single vertices, we see that $h(D) \geq h(H) \geq s+2t+1 = k$.

Case 2.2 p and k are odd.

In this case, $n-p$ is even. If $n-p \le 2(n-k)$, let $s=n-k$ and $t=k-\frac{n+1}{2}$. Then $\frac{k}{3} < s \le k-1$, $0 \le t < \frac{k}{3}$. Observe that s is even, $s+2t+1=k$ and $t \le s$. Let $G=I(k-1,s,t)$. Then G is a $(k-1)$-regular graph of order n. By Theorem 1.3, let M be a perfect matching and F be an anti-matching of size $\frac{n-p}{2}$ of $G[A,B]$, respectively. Let $H=G+F$. Then $H \in R[D]$. By contracting each of the edges of M into single vertices, we see that $h(H) \ge s+2t+1=k$. Hence $h(D) \ge k$.

So we may assume that $n-p \ge 2(n-k)+2$. Then $p \le n-2(n-k)-2 = 2k-n-2 < 2(n-k)$. Let $s=n-k-1$ and $t=k+1-\frac{n+1}{2}$. Then $\frac{k}{3} < s \le k-2$, $1 \le t < \frac{k}{3}$. Observe that s is odd, $s+2t+1=k+1$ and $t<s$ because $3n \ge 4k+4$. Let J be an Hamiltonian cycle of K_s and let E be a matching of size $\frac{s-1}{2}$ in J. Now let G be a graph of order n with $V(G)$ partitioned into A,B,C,D such that $G[A]=K_s$, $G[B]=K_s-E(J)+E$, $G[C]=K_t$, $G[D]=K_{t+1}$, $G[A,B]$ is a $(t+1)$-regular bipartite graph. Moreover, C is complete to $A \cup D$, D is complete to B, A is anticomplete to D, C is anticomplete to B. Let w be the vertex of degree $s-3$ in $G[B]$. Since $t \ge 1$, by Theorem 1.3, $G[A,B]$ contains two perfect matchings M_1 and M_2. Since $p < 2s$, there exists $F \subset M_1$ with $|F| = \frac{p-1}{2}$ and $w \notin V(F)$. Let $H=G-F$. Then $H \in R[D]$. By contracting each of the edges of M_2 into single vertices, we see that $h(D) \ge h(H) \ge s+2t+1=k+1$.

This completes the proof of Lemma 2.2. ■

Theorem 2.3 *Let $D=((k-1)^p,\ k^{n-p})$ be the degree sequence of a near k-regular graph on n vertices, where $0 \le p \le n-1$. Then*

$$h(D) \ge \begin{cases} k+1 & \text{if} \quad p=0 \quad \text{and} \quad n=k+1 \\ k+1 & \text{if} \quad p=0 \quad \text{and} \quad n \ge \frac{4k+4}{3} \\ k & \text{if} \quad p>0 \quad \text{and} \quad n \ge \frac{4k+4}{3} \\ k+1-\lceil \frac{p}{4} \rceil & \text{if} \quad p>0 \quad \text{and} \quad n=k+1 \\ \lfloor \frac{3n}{4} \rfloor & \text{if} \quad k+2 \le n < \frac{4k+4}{3}. \end{cases}$$

Proof. If $n=k+1$ and $p=0$, then $R[D]=\{K_{k+1}\}$ and $h(D)=k+1$. If $n=k+1$ and $p>0$, then p must be even and D has a unique realization $K_{k+1}-M$, where M is a matching of size $\frac{p}{2}$ of K_{k+1}. Clearly, $h(D) \ge (n-p)+\frac{p}{2}+\lfloor \frac{p}{4} \rfloor \ge k+1-\lceil \frac{p}{4} \rceil$. So we may assume that $n \ge k+2$. If $n=k+2$ and $p=0$, then k must be even and D has a unique realization K_n-M, where M is a perfect matching of K_n. It can be easily checked that $h(D)=\lfloor \frac{3n}{4} \rfloor$. Suppose $n=k+2$ and $p>0$. If k is odd, then n and p must be odd. Let J be a Hamilton cycle of K_n and let F be a matching of size $\frac{n-p}{2}$ of J. Let $G=K_n-E(J)+F$. Then $G \in R[D]$. Let $\{x_1,x_2,\dots,x_n\}$ be the vertices of J in order. Let $M=\{x_3x_n, x_5x_{n-2}, \dots, x_{\frac{n+1}{2}}x_{\frac{n+1}{2}+2}\}$ if $\frac{n-1}{2}$ is even; and $M=\{x_1x_{\frac{n+1}{2}+1}, x_3x_n, x_5x_{n-2}, \dots, x_{\frac{n-1}{2}}x_{\frac{n+1}{2}+3}\}$ if $\frac{n-1}{2}$ is odd. By contracting each of the edges of M into single vertices, we see that $h(D) \ge h(G) \ge \lfloor \frac{3n}{4} \rfloor$. If k is even, then n and p must be even. Let $x_1,x_2,\dots,x_{\frac{n}{2}},y_1,y_2,\dots,y_{\frac{n}{2}}$ be the n vertices of K_n. Let $G=K_n-E$, where $E=\{x_1y_1, x_2y_2, \dots, x_{\frac{n}{2}}y_{\frac{n}{2}}, y_1x_2, y_2x_3, \dots, y_{\frac{p}{2}}x_{\frac{p}{2}+1}\}$. By contracting each of the edges

$y_i y_{\lfloor \frac{n}{4} \rfloor + i}$, where $i = 1, 2, \ldots, \lfloor \frac{n}{4} \rfloor$, into single vertices, we see that $h(D) \geq h(G) \geq \lfloor \frac{3n}{4} \rfloor$. Thus we may assume that $n \geq k + 3$.

By Lemma 2.2, we may assume that $k + 3 \leq n < \frac{4k+4}{3}$. We next show that $h(D) \geq \lfloor \frac{3n}{4} \rfloor$. Note that $\overline{D}$ is the degree sequence of a near r-regular graph on n vertices, where $2 \leq r = n - 1 - k < \frac{k+1}{3}$. Since $3n < 4k + 4$, we have $4r = 4(n - k - 1) < n$. Let m, s be nonnegative integers so that $n = 4m + s$, where $0 \leq s \leq 3$. Clearly, $m \geq r$ because $n > 4r$. Assume that $p = 0$. For an integer $t > 0$, we denote by $B^r_{t,t}$ an r-regular bipartite graph with each partite of size t, and let M be a matching of $B^r_{t,t}$ of size $[\frac{r}{2}]$. Let B^* be obtained from $B^r_{m,m} - M$ by adding a new vertex v joining to each vertex of $V(M)$. Note that if n is odd, then r is even. Let

$$\overline{G} = \begin{cases} K^r_{m,m} \cup K^r_{m,m} & \text{if} \quad s = 0 \\ K^r_{m,m} \cup B^* & \text{if} \quad s = 1 \\ K^r_{m,m} \cup K^r_{m+1,m+1} & \text{if} \quad s = 2 \\ K^r_{m+1,m+1} \cup B^* & \text{if} \quad s = 3 \end{cases}$$

It can be easily checked that $G \in R[D]$ and G contains a clique minor of size at least $\lfloor \frac{3n}{4} \rfloor$. Thus $h(D) \geq \lfloor \frac{3n}{4} \rfloor$, as desired. So we may assume that $p > 0$. We consider the following two cases.

Case 1. n is even.

Then $n = 4m$ or $4m + 2$, and p must be even. Since $p \leq n - 1$, we have $p \leq n - 2$. Let H be a k-regular graph of order n with $V(H)$ partitioned into A, B, C, D such that $H[A] = K_m$, $H[B] = K_{\frac{n-2m}{2}}$, $H[C] = K_{\frac{n-2m}{2}}$, $H[D] = K_m$, $A \cup D$ is complete to $B \cup C$, $H[A, D]$ and $H[B, C]$ are $(m - r)$-regular and $(m + 1 - r)$-regular bipartite graphs, respectively. If $r < m$, let M and M' be perfect matchings of $H[A, D]$ and $H[B, C]$, respectively. Let $F \subseteq M \cup M'$ with $|F| = \frac{p}{2}$. If $r = m$ and $p \leq 2m + 2$, then $n = 4m + 2$. Thus $|B| = |C| = m + 1$. Let M be a perfect matching of $G[B, C]$. Let $F \subseteq M$ with $|F| = \frac{p}{2}$. In both cases, let $H^* = H - F$. Then $H^* \in R[D]$. By contracting each of the edges of a matching of size m in $H[A, B]$ into single vertices, we see that $h(D) \geq h(H^*) \geq \lfloor \frac{3n}{4} \rfloor$. So we may assume that $r = m$ and $p \geq 2m + 4$. Since $r = m$, we have $n = 4m + 2$ and so $k = 3m + 1$. Note that $n - p \leq (4m + 2) - (2m + 4) = 2m - 2$. Let $H = G(3m, m + 1, m)$, as defined in the proof of Lemma 2.2 (see Case 1). Let A and B be as given in the definition of $G(3m, m+1, m)$. Now let M and M' be a perfect matching and an anti-matching of size $\frac{n-p}{2}$ of $G(3m, m+1, m)[A, B]$, respectively. Let $H^* = H + M'$. Then $H^* \in R[D]$. By contracting each of the edges of M into single vertices, we have $h(D) \geq h(H^*) \geq \lfloor \frac{3n}{4} \rfloor$.

Case 2. n is odd.

In this case, $n = 4m + 1$ or $4m + 3$. Since $\sigma(D) = nk - p$ is even, we have

(b) p and k are either both even or both odd.

Let $s = k$ if k is even and $s = k - 1$ if k is odd. Then $n - 1 - s$ is even and let $r' = (n - 1) - 1 - s$. Then $r' = r - 1$ if $s = k$ and $r' = r$ if $s = k - 1$. Define $H(s)$ to be an $(s-1)$-regular graph of order $n-1$ with $V(H)$ partitioned into A, B, C, D such that $H(s)[A] = K_m$, $H(s)[B] = K_{\frac{n-1-2m}{2}}$, $H(s)[C] = K_{\frac{n-1-2m}{2}}$, $H(s)[D] = K_m$, $A \cup D$ is complete to $B \cup C$, $H(s)[A, D]$ and $H(s)[B, C]$ are $(m - r' - 1)$-regular and $(\frac{n-1-2m}{2} - r' - 1)$-regular bipartite graphs, respectively. Note that $|B| = |C| = m$ if $n = 4m+1$ and $|B| = |C| = m+1$ if $n = 4m+3$. Let F be an anti-matching of size $\frac{n-1-s}{2}$ of $H(s)[A, D]$. Let $H^*(s)$ be the graph obtained from $H(s) + F$ by joining a new vertex w to $V(H) \backslash V(F)$. Then $H^*(s)$ is s-regular.

If k is odd, by (b), $n - p$ is even. Note that $s = k - 1$ and $r' = r \geq 2$. Let $X = H^*(s)$, as defined above. Then X is $(k-1)$-regular. Let M and M' be two perfect anti-matchings of $H(s)[A, D]$ and $H(s)[B, C]$, respectively. Let $F \subseteq M \cup M'$ with $|F| = \frac{n-p}{2}$. Then $X + F \in R[D]$. If k is even, by (b), p is even. Note that $s = k$ and $r' = r - 1$. Thus $m - r' = m - (r-1) = m + 1 - r \geq 1$ and $\frac{n-1-2m}{2} - r' \geq m - r' \geq 1$. Let $Y = H^*(s)$, as defined above. Then Y is k-regular. Let M and M' be perfect matchings of $H(s)[A, D]$ and $H(s)[B, C]$, respectively. Let $F \subseteq M \cup M'$ with $|F| = \frac{p}{2}$. One can easily check that $Y - F \in R[D]$. In both cases, let M^* be a matching of size m of $H^*(s)[A, B]$. Let $w' \in B \backslash V(M^*)$ if $|B| = m + 1$. Now by contracting each of the edges of M^* (and $w'w$ if $|B| = m+1$) into single vertices, we see that $h(D) \geq \lfloor \frac{3n}{4} \rfloor$.

This completes the proof of Theorem 2.3. ∎

We are now ready to prove the main result.

Theorem 2.4 *Let $D = ((k-1)^p, k^{n-p})$ be the degree sequence of a near k-regular graph on n vertices, where $0 \leq p \leq n-1$. Then $h(D) \geq \chi(D)$.*

Proof. By Theorem 1.2 and Theorem 2.3, we may assume that $n = k+1$ and $h(D) \geq k+1-\lceil \frac{p}{4} \rceil$ or $k+2 \leq n < \frac{4k+4}{3}$ and $h(D) \geq \lfloor \frac{3n}{4} \rfloor$. In the first case, p is even and D has a unique realization $K_{k+1} - M$, where M is a matching of size $\frac{p}{2}$. Clearly, $\chi(D) = k + 1 - \frac{p}{2} \leq h(D)$. It remains to show that $\chi(D) \leq \lfloor \frac{3n}{4} \rfloor$ when $k + 2 \leq n < \frac{4k+4}{3}$. Let $G \in R[D]$ and let M be a maximum anti-matching of G. Then $G \backslash V(M)$ is a complete subgraph. Thus $\chi(G) \leq |G \backslash V(M)| + |M| = (n - 2|M|) + |M| = n - |M|$. It suffices to show that $|M| \geq \frac{n}{4}$. If $p = 0$, then G is k-regular and $\overline{G}$ is r-regular, where $r = n - 1 - k \geq 1$. By Theorem 1.4, $\chi'(\overline{G}) \leq r + 1$, we have $\frac{rn}{2} = |E(\overline{G})| \leq (r+1)|M|$, which yields $|M| \geq \frac{n}{4}$. Assume that $p > 0$. Then $\overline{G}$ contains $n - p$ vertices of degree $r = n - 1 - (k-1) = n - k$ and p vertices of degree $n - 1 - k$. Let $r = n - k$. Then $r \geq 2$. If $r \geq 3$, by Theorem 1.4 again, $\chi'(\overline{G}) \leq r+1$, we have $\frac{(r-1)n}{2} < |E(\overline{G})| \leq (r+1)|M|$. It follows that $|M| \geq \frac{n}{4}$. So we may assume that $r = 2$. Then $\overline{G}$ consists of disjoint unions of cycles and paths. One can easily check that $\overline{G}$ contains a matching of size at least $\frac{n}{3}$. Thus $|M| \geq \frac{n}{4}$ and so $\chi(G) \leq n - |M| \leq \frac{3n}{4}$, as desired. Consequently, $\chi(D) \leq \lfloor \frac{3n}{4} \rfloor$ by the arbitrary choice of $G \in R[D]$.

This completes the proof of Theorem 2.4. ∎

Acknowledgement

The authors would like to thank the referees for their helpful comments.

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