The Erdös-Jacobson-Lehel Conjecture on Potentially P_k-Graphic Sequence is true *

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Abstract In this paper, we study a variation of the classical Tur*á*n extremal problem. A simple graph G of order n is said to have property P_k if it contains a clique of size $k+1$ as its subgraph. An n-term nonincreasing nonnegative integer sequence $\pi = (d_1, d_2, \cdots, d_n)$is said to be graphic if it is the degree sequence of a simple graph G of order n and such graph G is referred to a realization of π. A graphic sequence π is said to be potentially P_k-graphic if it has a realization G having property P_k. Erdös, Jacobson and Lehel[1] raised the following problem: determine the smallest positive even number $\sigma(k,n)$ such that every n-term graphic sequence $\pi = (d_1, d_2, \cdots, d_n)$ without zero terms and with degree sum $\sigma(\pi)$ $=d_1 + d_2 + \cdots + d_n$ at least $\sigma(k,n)$ is potentially P_k-graphic. They gave a lower bound of $\sigma(k,n)$ by the example $\pi_0 = ((n-1)^{k-1}, (k-1)^{n-k+1})$, i.e. , $\sigma(k,n) \geq (k-1)(2n-k)+2$, and conjectured that the lower bound is the exact value of $\sigma(k,n)$. They also proved the conjecture is true for $k=2$ and $n \geq 6$. In this paper, we prove that the conjecture is true for $k \geq 5$ and $n \geq \binom{k}{2}+3$. We once proved the conjecture is true[2),3)] for $k=3$ and $n \geq 8$ and for $k=4$ and $n \geq 10$, so the conjecture has been proved positive.

Key words: graph graphic sequence off-diagonal leftmost matrix potentially P_k-graphic sequence

An n-term nonincreasing nonnegative integer sequence $\pi = (d_1, d_2, \cdots, d_n)$ is said to be graphic if it is the degree sequence of a simple graph G of order n and such

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graph is referred to a realization of π. A graph G is said to have property P_k if it has a complete subgraph K_{k+1} of order $k+1$[1]. A graph sequence π is potentially (res. forcibly) P_k-graphic if it has a realization G having property P_k (res.all of its realizations have property P_k). The degree sequence is one of the basic subjects in the graph theory. As to its progress, please refer to the summarized essays S.R.Rao[2] and Li Jiong-Sheng[3].

It is well-known that the classical Turán extremal problem is to determine the smallest positive integer $ex(k,n)$ such that each graph G of order n with edge number $\varepsilon(G) \geq ex(k,n)$ contains a complete subgraph of order $k+1$. The number $ex(k,n)$ is called the Turán number. The classical Turán theorem determines the Turán number ([4],Ch6). As Bollobás pointed out in the preface of the masterpiece "Extremal Graph Theory" :"Extremal graph theory, in its strictest sense, is a branch of graph theory developed and loved by Hungarians. Its study, as a subject in its own right, was initiated by Turán in 1940, although a special case of his theorem and several other extremal results had been proved many years earlier. The main exponent has been Paul Erdös who, through his many papers and lectures, as well as uncountably many problems, has virtually created the subject." It serves to show that the Turán theorem is one of the basic theorems in extremal graph theory and that the extremal theory of graphs is carried out around it and also to show Erdös' historical position in the extremal graph theory.

In the view of the theory of graphic sequences, the Turán number $ex(k,n)$is the smallest positive integer such that each graphic sequence $\pi = (d_1, d_2, \cdots, d_n)$with degree sum $\sigma(\pi) = d_1 + d_2 + \cdots + d_n \geq 2ex(k,n)$ is forcibly P_k-graphic. Erdös, Jacobson and Lehel[1] considered a variation of the classical Turán extremal problem: determine the smallest positive even number $\sigma(k,n)$ such that each n-term graphic sequence $\pi = (d_1, d_2, \cdots, d_n)$ without zero terms and with degree sum $\sigma(\pi) \geq \sigma(k,n)$ is potentially P_k-graphic. They showed that the graphic sequence $D_k = ((n-1)^{k-1}, (k-1)^{n-k+1})$ is not potentially P_k-graphic, where $(n-1)^{k-1}$ means $n-1$ repeats $k-1$ times in π. So $\sigma(k,n) \geq (k-1)(2n-k)+2$. In [1], they also pointed out: "We feel that these are the extremal examples, that is, any degree sequence givening more edges than D_k is potentially P_k-graphical." In other words, [1] raised the following conjecture(abbrev. EJLC): $\sigma(k,n) = (k-1)(2n-k)+2$. As to the k and n such that EJLC holds, [1] didn't say any words. In the conclusion of the paper [1)] of Gould, Jacobson and Lehel, they posed clearly the conjecture: "Conjecture: For n sufficiently large, $\sigma(k,n) = (k-1)(2n-k)+2$." This shows that we only need to prove the EJLC is positive for any given $k \geq 2$ and n which is large enough. Erdös, Jacobson and Lehel[1] also proved the following

Theorem A. Let $n \geq 6$ and $\pi = (d_1, d_2, \cdots, d_n)$be a graphic sequence, where $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$. If $\sigma(\pi) \geq 2n$, then π is potentially P_2-graphic. In other words, $\sigma(2,n) = 2n$ for $n \geq 6$.

Recently, J.S.Li and Z.X.Song[2)] proved the following result.

Theorem 0.1. $\sigma(k, 2k+1) = 4k^2 - 2k$.

J.S.Li and Z.X.Song[2)] and Gould, Jacobson and Lehel[1)] proved the following result independently.

Theorem 0.2. $\sigma(3,n) = 4n-4$ for $n \geq 8$.

J.S.Li and Z.X.Song[3)] also proved

Theorem 0.3. For $n \geq 10$, $\sigma(4,n) = 6n - 10$.

The above results show that the EJLC is true for $2 \leq k \leq 4$ and $n \geq 2k+2$. The purpose of this paper is to prove the EJLC is true for $k \geq 5$ and $n \geq \binom{k}{2}+3$.

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1) Gould R J, Jacobson M S, Lehel J. Potentially G-graphical degree sequences, to appear.

2) Li Jiong-Sheng, Song Zi-Xia. An extremal problem on the potentially P_k-graphical sequence. In the International Symposium on Combinatorics and Applications. June 28-30,1996, Tianjin, W.Y.C.Chen et.al. ed,Nankai University, Tianjin, 1996, 266-276.

3) Li Jiong-Sheng, Song Zi-Xia. The smallest degree sum that yields potentially P_k-graphical sequence, submitted.

1 Preparation

We need the following terms and notations. Let $\pi = (d_1, d_2, \cdots, d_n)$ be an integer sequence with $n-1 \geq d_1 \geq d_2 \geq \cdots \geq d_n \geq 0$. The set of all such sequences is denoted by NS_n. For a given $\pi = (d_1, d_2, \cdots, d_n) \in NS_n$, let $\sigma_i(\pi) = d_1 + d_2 + \cdots + d_i$ for each $1 \leq i \leq n$ and $\sigma_n(\pi) = \sigma(\pi)$. In addition, let $f(\pi) = max\{i : d_i \geq i, 1 \leq i \leq n\}$. $\sigma(\pi)$ and $f(\pi)$ are called the degree sum and trace of π, respectively. Let's define an $n \times n$ matrix $\overline{A(\pi)} = (a_{ij})$ as follows:
For $1 \leq i \leq f(\pi)$,

$$a_{ij} = \begin{cases} 1, & if\ 1 \leq j \neq i \leq d_i + 1, \\ 0, & otherwise, \end{cases}$$

and for $f(\pi) + 1 \leq i \leq n$,

$$a_{ij} = \begin{cases} 1, & if\ 1 \leq j \leq d_i, \\ 0, & otherwise. \end{cases}$$

Then the matrix $\overline{A(\pi)}$ is called the off-diagonal leftmost matrix of π. Clearly, the row sum vector of $\overline{A(\pi)}$ is π. The column sum vector of $\overline{A(\pi)}$ is called the corrected conjugate vector of π, and denoted by $\overline{\pi}$. Obviously,$\sigma(\pi) = \sigma(\overline{\pi})$. The set of all n-term graphic sequences are denoted by GS_n. The following is a creteria for a nonincreasing sequence being graphic.

Theorem B. (Berge[5]) Let $\pi = (d_1, d_2, \cdots, d_n) \in NS_n$ with even $\sigma(\pi)$. Then $\pi \in GS_n$ if and only if for each $t = 1, 2, \cdots, n, \sigma_t(\pi) \leq \sigma_t(\overline{\pi})$, or in equivalent words, if and only if for each $t = 1, 2, \cdots, f(\pi)$, $\sigma_t(\pi) \leq \sigma_t(\overline{\pi})$.

Let $\pi = (d_1, d_2, \cdots, d_n) \in NS_n$. Denote

$$\pi' = \begin{cases} (d_1 - 1, \cdots, d_{k-1} - 1, d_{k+1} - 1, \cdots, d_{d_k+1} - 1, d_{d_k+2}, \cdots, d_n), & if\ d_k \geq k, \\ (d_1 - 1, \cdots, d_{d_k} - 1, d_{d_k+1}, \cdots, d_{k-1}, d_{k+1}, \cdots, d_n), & if\ d_k \leq k - 1. \end{cases}$$

Then π' is the residual sequence obtained by laying off d_k from π.

Theorem C.(Kleitman and D.L.Wang[6]) Let $\pi \in NS_n$. Then $\pi \in GS_n$ if and only if $\pi' \in GS_{n-1}$.

Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and let $\pi =$

$(d_1, d_2, \cdots, d_n)$ be the degree sequence of G, where d_i is the degree of v_i and π is not necessarily nonincreasing. Then G is said to have property A_k if its subgraph induced by $v_1, v_2, \cdots, v_{k+1}$ is complete, i.e., if its first $k+1$ vertices form a clique of size $k+1$. A degree sequence π is said to be potentially A_k-graphic if there exists a graph G having property A_k realizes π. A.R.Rao[7] proved the following

Theorem D. Let π be the degree sequence of graph G. Then π is potentially P_k-graphic if and only if π is potentially A_k-graphic.

Let $\pi = (d_1, d_2, \cdots, d_n)$be a sequence of nonnegative integers. For any given s and t, $0 \le s \le k+1$ and $0 \le t \le n-k-1$, denote
$L(s,t)= \sum\limits_{i=1}^{s} d_i + \sum\limits_{j=1}^{t} d_{k+1+j}$,
$R(s,t)= (s+t)(s+t-1) + \sum\limits_{i=s+1}^{k+1} min\{s+t, d_i-k+s\} + \sum\limits_{j=k+2+t}^{n} min\{s+t, d_j\}$.

A.R.Rao[4] also gave a creteria for a sequence π being potentially A_k graphic as follows.

Theorem E. Let $\pi = (d_1, d_2, \cdots, d_n)$be a sequence of nonnegative integers in which $d_1 \ge d_2 \ge \cdots \ge d_{k+1}$ and $d_{k+2} \ge d_{k+3} \ge \cdots \ge d_n$. Then π is potentially A_k-graphic if and only if the following conditions hold:

(1) $d_{k+1} \ge k$,

(2) $\sigma(\pi)$ is even,

(3) for any s and t, $0 \le s \le k+1$, $0 \le t \le n-k-1$,

$$L(s,t) \le R(s,t). \tag{1}$$

But we have to point out with regret that Theorem E has never being published though it is mentioned in A.R.Rao[7]. We have carefully looked through his original proof and make sure that his proof is right. Recently we were informed that K*é*zdy and Lehel [5] have given a proof of the Theorem E.

In order to prove the main results of this paper, we will use double induction on

[4] Rao A R. An Erdös-Gallai type result on the clique number of a realization of a degree sequence, unpublished.

[5] K*é*zdy A, Lehel J. Degree sequences of graphs with prescribed clique size, to appear.

k and n, Kleitman and Wang's laying off technique, Theorems D and E. We will also use the following simple fact repeatedly: Let $\pi' = (d'_1, d'_2, \cdots, d'_{n-1})$ be the residual sequence obtained by laying off d_t from π. If π' is potentially A_{k-1}-graphic and the first k terms $d'_1, \cdots, d'_k$ of π' are obtained by subtracting one from the k terms of π respectively, then π is potentially P_k-graphic. In addition, we need the following facts.

Theorem 1.1. Let $\pi = (d_1, d_2, \cdots, d_n) \in GS_n$ with $d_{k+1} \geq k$ and

$$n-2 \geq d_1 \geq \cdots \geq d_k = d_{k+1} = \cdots = d_{d_1+2} \geq \cdots \geq d_n \geq k-1.$$

If $n = 2k+2$ and $d_n \geq k$, or $n \geq 2k+3$ and $d_n \geq k-1$, then π is potentially P_k-graphic.

Proof. By Theorem E, we only need to verify that (1) holds for any s and t, $0 \leq s \leq k+1$ and $0 \leq t \leq n-k-1$. We consider two cases as follows.

Case 1: $d_k \leq s+t-1$. Suppose $n \geq 2k+2$ and $d_n \geq k-1$. If $s \geq k$, then $s+t \geq s \geq k$ and $d_i - k + s \geq (d_k - k) + s \geq s \geq k$ for $1 \leq i \leq k+1$. Hence $min\{s+t, d_i - k + s\} \geq k$ for $1 \leq i \leq k+1$. In addition, $s+t > d_k \geq \cdots \geq d_n \geq k-1$, hence $min\{s+t, d_j\} \geq d_j \geq k-1$ for $k \leq j \leq n$. Therefore

$$\begin{aligned} R(s,t) &\geq (s+t)(s+t-1) + (k+1-s)k + (n-k-1-t)(k-1) \\ &= (k-1)(n-2) + (s+t+1-k)(s+t-1) + (2k-s) \\ &\geq (k-1)(n-2) + (s-k+1)(s+t-1) + t(s+t-1) \\ &\geq (k-1)d_1 + (s-k+1)d_k + td_k \geq L(s,t). \end{aligned}$$

If $0 \leq s \leq k-1$,then $d_i - k + s \geq (d_k - k) + s \geq s$ for $1 \leq i \leq k+1$ and $s+t > d_k \geq \cdots \geq d_n \geq k-1 \geq s$. Hence

$$\begin{aligned} R(s,t) &\geq (s+t)(s+t-1) + (k+1-s)s + (n-k-1-t)s \\ &= s(n-1) + t(s+t-1) \\ &> sd_1 + td_k \geq L(s,t). \end{aligned}$$

So (1) holds for $d_k \leq s+t-1$ if $n \geq 2k+2$ and $d_n \geq k-1$.

Case 2: $d_k \geq s+t$. In this case, $d_k = d_{k+1} = \cdots = d_{d_1+2} \geq s+t$. Let $n \geq 2k+2$ and $d_n \geq k-1$. If $d_k \geq t+k$,then $d_i - k + s \geq (d_k - k) + s \geq s+t$ for $1 \leq i \leq k+1$. Hence

$$\begin{aligned} R(s,t) &\geq (s+t)(s+t-1) + \sum_{i=s+1}^{k+1} min\{s+t, d_i - k + s\} + \sum_{j=k+2+t}^{d_1+2} min\{s+t, d_j\} \\ &= (s+t)(s+t-1) + (k+1-s)(s+t) + (d_1 + 2 - k - 1 - t)(s+t) \\ &= (s+t)(d_1+1) \geq sd_1 + td_k \geq L(s,t). \end{aligned}$$

If $s+t \leq d_k \leq t+k-1$, then $0 \leq s \leq k-1$ and $d_k - k + s < s+t$. Moreover $d_k - k + s \leq d_i - k + s$ for $1 \leq i \leq k+1$, so $min\{s+t, d_i - k + s\} \geq d_k - k + s$

for $1 \leq i \leq k+1$. Denote $d_k = t + m$, where $0 \leq s \leq m \leq k-1$. Then $s + t = d_k - (m-s) \geq k - m + s$. If $d_n \geq k$, then $d_n \geq k - (m-s)$. Hence

$$\begin{aligned} R(s,t) &\geq (s+t)(s+t-1) + (k+1-s)(d_k - k + s) + (n-k-1-t)(k-m+s) \\ &= s(n-2) + td_k + (k-m)(n-2k-2) + s(k+1-m) \qquad (2) \\ &\geq s(n-2) + td_k \geq sd_1 + td_k \geq L(s,t). \end{aligned}$$

So (2) holds for $n \geq 2k+2$ and $d_n \geq k$.

Combining Cases 1 and 2, we have proved that (1) holds for $n \geq 2k+2$ and $d_n \geq k$.

Now suppose $n \geq 2k+3$ and $d_n = k-1$. Then $d_1 \leq n-3$ since $d_{d_1+2} = d_{k+2} \geq k > k-1 = d_n$. In the proof of Case 2, if $s+t < d_k \leq t+k-1$,then $0 \leq s \leq k-1$ and the m in $d_k = t+m$ satisfies $0 \leq s < m \leq k-1$. So $d_n = k-1 \geq k-(m-s)$ still holds. Hence, (2) holds for $s+t < d_k \leq t+k-1$ if $n \geq 2k+3$ and $d_n = k-1$. If $s+t = d_k \leq t+k-1$,then $0 \leq s \leq k-1$ and

$$\begin{aligned} R(s,t) &\geq (s+t)(s+t-1) + (d_k - k + s)(k+1-s) + (n-k-1-t)(k-1) \\ &= (s+t)(s+t-1) + (s+t-k+s)(k+1-s) + (n-k-1-t)(k-1) \\ &= s(n-3) + t(s+t) + (n-2k-3)(k-1-s) + s(k-s) + (s+t-2) \\ &\geq sd_1 + td_k \geq L(s,t). \end{aligned}$$

Hence, (1)holds for $d_k \geq s+t$ if $n \geq 2k+3$ and $d_n \geq k-1$. □

Theorem 1.2. Let $n \geq 2k+2$ and $\pi = (d_1, d_2, \cdots, d_n) \in GS_n$ where $d_{k+1} \geq k$,

$$n-3 \geq d_1 \geq \cdots \geq d_k = d_{k+1} = \cdots = d_{d_1+2} = \cdots = d_p > d_{p+1} \geq \cdots \geq d_n, \qquad (3)$$

$1 \leq d_n \leq k-1$ and $d_1 + d_n \geq n-2$. Denote $\pi' = (d_2 - 1, \cdots, d_s - 1, d_{s+1} - 2, \cdots, d_p - 2, d_{p+1} - 1, \cdots, d_{n-1} - 1)$, where $s \geq k+1$, then π' is graphic.

Proof. Denote $\pi' = (d'_1, d'_2, \cdots, d'_{n-2})$. Clearly $n-4 \geq d_1 - 1 \geq d'_1 \geq d'_2 \geq \cdots \geq d'_{n-2}$. In order to prove π' is graphic, we only need to verify by Theorem B that

$$\sigma_t(\pi') \leq \sigma_t(\overline{\pi'}) \qquad (4)$$

for each t,$1 \leq t \leq f(\pi')$.

First, $f(\pi) = d_k$ because $d_k = \cdots = d_{d_k} = \cdots = d_p \geq k$. So $f(\pi') \leq f(\pi) - 1 = d_k - 1$ by the definition of π'.

Next, from (3), the off-diagonal leftmost matrix $\overline{A(\pi)}$ of π has the following form:

	1	2	$\dots$	$k-1$	k	$k+1$	$\cdots$	d_k-1	d_k	d_k+1	d_k+2	$\cdots$	$n-3$	$n-2$	$n-1$	n
1	0	1	$\cdots$	1	1	1	$\cdots$	1	1	1	$*$	$\cdots$	$*$	$*$	0	0
2	1	0	$\cdots$	1	1	1	$\cdots$	1	1	1	$*$	$\cdots$	$*$	$*$	0	0
$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$					
$k-1$	1	1	$\cdots$	0	1	1	$\cdots$	1	1	1	$*$					
k	1	1	$\cdots$	1	0	1	$\cdots$	1	1	1	0					
$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$					
d_k-1	1	1	$\cdots$	1	1	1	$\cdots$	0	1	1	0					
d_k	1	1	$\cdots$	1	1	1	$\cdots$	1	0	1	0					
d_k+1	1	1	$\cdots$	1	1	1	$\cdots$	1	1	0	0					
$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$					
d_1+2	1	1	$\cdots$	1	1	1	$\cdots$	1	1	0	0					
$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$					
p	1	1	$\cdots$	1	1	1	$\cdots$	1	1	0						
$p+1$	1	1	$\cdots$	1	1	1	$\cdots$	$*$	0							
$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$					
$n-1$	1	1	$\cdots$	1	1	$*$										
n	1	1	$\cdots$	$*$	0											

While the off-diagonal leftmost matrix $\overline{A(\pi')}$ of π' is obtained from $\overline{A(\pi)}$ by deleting the first and n-th rows and columns and deleting the $p-s$ ones of the d_k-th column from the p-th row to $(s+1)$-th row if $s \geq d_k+1$ while if $k+1 \leq s \leq d_k$,deleting $p-d_k$ ones of the d_k-th column from the p-th row to (d_k+1)-th row, and then deleting $d_k - s$ ones of the (d_k+1)-th column from the d_k-th row to $(s+1)$-th row. Hence,$\overline{d_i'} \geq p-2 \geq d_1$ for $1 \leq i \leq d_k - 2$ and $\overline{d'_{d_k-1}} \geq d_k - 2$. So $\sigma_t(\overline{\pi'}) \geq td_1 \geq \sigma_t(\pi')$ for $1 \leq t \leq d_k - 2$. Notice that $d'_{d_k-1} \leq d_{d_k} - 1 = d_k - 1$. Hence, $\sigma_{d_k-1}(\overline{\pi'}) = \sigma_{d_k-2}(\overline{\pi'}) + \overline{d_{d_k-1}} \geq (d_k-2)d_1 + (d_k-2) \geq \sigma_{d_k-1}(\pi')$. This shows that (4) holds. $\square$

2 Main Results

We need following lemmas.

Lemma 2.1. If $k \geq 5$, $\pi = (d_1, d_2, \cdots, d_{2k+2}) \in GS_{2k+2}$, and $\sigma(\pi) \geq 4k^2 - 4k$, then $d_{k+1} \geq k$ and $d_{2k} \geq 2$.

Proof. If $d_{k+1} \leq k-1$ then we obtain from the off-diagonal leftmost matrix $\overline{A(\pi)}$ of π that $\overline{d_i} \leq 2k+1$ for $1 \leq i \leq k-1$ and $\overline{d_k} \leq k-1$. Hence by Theorem B,

$$\begin{aligned} 4k^2-4k \leq \sigma(\pi) &\leq \sigma_k(\pi)+(d_{k+1}+\cdots+d_{2k+2}) \\ &\leq \sigma_k(\overline{\pi})+(d_{k+1}+\cdots+d_{2k+2}) \\ &\leq \sigma_{k-1}(\overline{\pi})+\overline{d_k}+(d_{k+1}+\cdots+d_{2k+2}) \\ &\leq (k-1)(2k+1)+(k-1)+(k-1)(k+2) \\ &= (k-1)(3k+4)=3k^2+k-4, \end{aligned}$$

a contradiction, thus $d_{k+1} \geq k$.

If $d_{2k}=d_{2k+1}=d_{2k+2}=1$, then from the off-diagonal leftmost matrix$\overline{A(\pi)}$ of π, we know $\overline{d_1}=2k+1$ and $\overline{d_i} \leq 2k-2$ for $2 \leq i \leq 2k+2$. By Theorem B,

$$\begin{aligned} 4k^2-4k \leq \sigma(\pi) &= \sigma_{2k-1}(\pi)+d_{2k}+d_{2k+1}+d_{2k+2} \\ &\leq \sigma_{2k-1}(\overline{\pi})+3 \\ &\leq 2k+1+(2k-2)(2k-2)+3 \\ &= 4k^2-6k+8, \end{aligned}$$

i.e., $k \leq 4$, a contradiction. □

Remark: In the similar way, we can prove that $d_{k+1} \geq k$ if $\pi=(d_1,d_2,\cdots,d_n) \in GS_n$ and $\sigma(\pi) \geq (k-1)(2n-k)+2$.

Lemma 2.2. Let $\pi \in GS_{2k+2}$ without zero terms and $\sigma(\pi) \geq 4k^2-4k$. If $k=5$, or $\sigma(k-1,2k) \leq 4k^2-12k+8$ and $\sigma(k-1,2k+1) \leq 4k^2-10k+2$ for $k \geq 6$, then π is potentially P_k- graphic.

Proof. By Lemma 2.1, $d_1 \geq d_{k+1} \geq k$. Denote $\pi'=(d_2-1,\cdots,d_{d_1+1}-1,d_{d_1+2},\cdots,d_{2k+2})$. Then by Theorem C, π' is graphic and $\sigma(\pi')=\sigma(\pi)-2d_1 \geq 4k^2-4k-2(2k+1)=4k^2-8k-2$. By Lemma 2.1, $d_{2k} \geq 2$, i.e., π has at most two ones. So π' has at most two zeros. The sequence obtained by deleting the zero terms of π' is denoted by π''. Then $\sigma(\pi'')=\sigma(\pi') \geq 4k^2-8k-2$ and $\pi'' \in GS_{2k+1} \bigcup GS_{2k} \bigcup GS_{2k-1}$. If $k=5$, $\sigma(\pi'') \geq 58$ and $\pi'' \in GS_9 \bigcup GS_{10} \bigcup GS_{11}$. By Theorem 0.1, $\sigma(4,9)=56$. By Theorem 0.3, $\sigma(4,10)=50$ and $\sigma(4,11)=56$, therefore, $\sigma(\pi'') \geq max\{\sigma(4,9),\sigma(4,10),\sigma(4,11)\}$. If $k \geq 6$, by Theorem 0.1, $\sigma(k-1,2k-1)=4k^2-10k+6$. By the assumption, we have $\sigma(\pi'') \geq max\{(\sigma(k-1,2k-1),\sigma(k-1,2k),\sigma(k-1,2k+1)\}$. Therefore π'' is potentially A_{k-1}-graphic. Hence, if π' has zero terms, then π is potentially P_k-graphic. Suppose π' has no zero terms. If $d_1=2k+1$ or there exists t, $k+1 \leq t \leq d_1+1$, such that $d_t > d_{t+1}$, then the first k largest terms of π' are obtained by subtracting one from k terms of π, respectively, therefore π is potentially P_k-graphic. Hence we may assume

$$2k \geq d_1 \geq \cdots \geq d_k \geq d_{k+1} = \cdots = d_{d_1+1} = d_{d_1+2} \geq \cdots \geq d_{2k+2}.$$

If $d_k > d_{k+1}$,the sequence obtained by laying off $d_{d_1+1} = l$ from π is denoted by π'''. By Theorem C, $\pi''' \in GS_{2k+1}$. Moreover π''' has no zero terms and $\sigma(\pi''') \geq 4k^2 - 8k - 2$. By the assumption, $\sigma(\pi''') \geq \sigma(k-1, 2k+1)$. So π''' is potentially A_{k-1}-graphic, therefore π is potentially P_k-graphic. Hence we may assume

$$2k \geq d_1 \geq \cdots \geq d_k = d_{k+1} = \cdots = d_{d_1+2} \geq \cdots \geq d_{2k+2}.$$

If $d_{2k+2} \geq k$, then by Theorem 1.1, π is potentially P_k-graphic. Hence, assume $1 \leq d_{2k+2} \leq k-1$. Since $d_{d_1+2} = d_{k+1} \geq k > k-1 \geq d_{2k+2}$, we have $d_1 + 2 \leq 2k+1$, i.e., $d_1 \leq 2k-1$. If $d_1 \leq 2k-4$, then

$$4k^2 - 4k \leq \sigma(\pi) \leq (2k+1)(2k-4) + (k-1) = 4k^2 - 5k - 5,$$

a contradiction. Therefore $2k-3 \leq d_1 \leq 2k-1$. We consider two cases as follows.

Case 1: $d_1 + d_{2k+2} \geq 2k$. Denote

$$\begin{aligned} &2k-1 \geq d_1 \geq \cdots \geq d_k = d_{k+1} = \cdots = d_{d_1+1} \\ &= d_{d_1+2} = \cdots = d_p > d_{p+1} \geq \cdots \geq d_{2k+2}, \end{aligned}$$

where $d_1 + 2 \leq p \leq 2k+1$ and denote

$$\pi' = (d_2 - 1, \cdots, d_s - 1, d_{s+1} - 2, \cdots, d_p - 2, d_{p+1} - 1, \cdots, d_{2k+1} - 1),$$

where $d_1 + d_{2k+2} = 2k + p - s$. Since $2k + p - s = d_1 + d_{2k+2} \leq 2k - 1 + k - 1 = 3k - 2$, we have $s \geq p + 2 - k \geq d_1 + 4 - k \geq 2k - 3 + 4 - k = k + 1$. By Theorem 1.2, π' is graphic. Notice that $d_p - 2 = d_{k+1} - 2 \geq k - 2 > 0$. In addition, by Lemma 2.1, $d_{2k} \geq 2$. Therefore π' has at most a zero. If π' has exact a zero term, then $d_{2k+1} = d_{2k+2} = 1$. Since $d_1 + d_{2k+2} = d_1 + 1 \geq 2k$, we have $d_1 \geq 2k-1$, therefore $d_1 = 2k-1$. Hence $\pi'' = (d_2 - 1, d_3 - 1, \cdots, d_{2k} - 1, d_{2k+1}, d_{2k+2}) \in GS_{2k+1}$, has no zero terms and $\sigma(\pi'') = \sigma(\pi) - 2(2k-1) \geq 4k^2 - 8k + 2$. If $k = 5$, then by Theorem 0.3, $\sigma(4, 11) = 56 < 58 = 4 \times 5^2 - 8 \times 5 - 2 \leq \sigma(\pi'')$, so π'' is potentially A_4-graphic. Therefore, π is potentially P_5-graphic. If $k \geq 6$, then by the assumption, $\sigma(k-1, 2k+1) \leq 4k^2 - 10k + 2 < 4k^2 - 8k - 2 \leq \sigma(\pi'')$, therefore π'' is potentially A_{k-1}-graphic, so π is potentially P_k-graphic. If π' has no zero terms, then $\sigma(\pi') = \sigma(\pi) - 2(d_1 + d_{2k+2}) \geq 4k^2 - 4k - 2(3k-2) = 4k^2 - 10k + 4$. If $k = 5$, by Theorem 0.3, $\sigma(4, 10) = 50 < 54 = 4k^2 - 10k + 4 \leq \sigma(\pi')$. If $k \geq 6$, by the assumption, $\sigma(k-1, 2k) \leq 4k^2 - 12k + 8 < 4k^2 - 10k + 4 \leq \sigma(\pi')$. Therefore π' is potentially A_{k-1}-graphic. Notice that π' is obtained by twice deleting term operations from π:

one is to lay off d_1 from π and the other is to subtract one respectively, from the $(2k+1-d_{2k+2})$-th term to the term before the last of the residual sequence and then delete the last term. Thus π is potentially P_k-graphic.

Case 2: $d_1+d_{2k}\le 2k-1$. In this case, $1\le d_{2k+2}\le 2$ because $2k-3+d_{2k+2}\le d_1+d_{2k+2}\le 2k-1$. If $d_{2k+2}=2$, then $d_1=2k-3$. Therefore $4k^2-4k\le \sigma(\pi)\le (2k+1)(2k-3)+2=4k^2-4k-1$, a contradiction. Hence $d_{2k+2}=1$. So $2k-3\le d_1\le 2k-2$. If $d_1=2k-3$, then $4k^2-4k\le\sigma(\pi)\le(2k+1)(2k-3)+1=4k^2-4k-2$, a contradiction. Hence $d_1=2k-2$ and $d_{d_1+2}=d_{2k}=d_{k+1}$. If $d_{k+1}\le 2k-4$, then $4k^2-4k\le\sigma(\pi)\le(k-1)(2k-2)+(k+2)(2k-4)+1=4k^2-4k-5$, a contradiction. Hence $2k-3\le d_{k+1}\le 2k-2$. If $d_{k+1}=2k-2$, then $\pi=((2k-2)^{2k},d_{2k+1},1^1)$. Because $\pi\in GS_{2k+2}$,d_{2k+1} is odd. Therefore $d_{2k+1}\le 2k-3$. Hence the residual sequence obtained by laying off $d_1=2k-2$ from π is $\pi'=((2k-2)^1,(2k-3)^{2k-2},d_{2k+1},1^1)$, and then the residual sequence obtained by laying off $d_{2k+2}=1$ from π' is $\pi''=((2k-3)^{2k-1},d_{2k+1})$. By Theorem C, $\pi''\in GS_{2k}$ and has no zero terms. In addition, $\sigma(\pi'')=\sigma(\pi)-2(2k-1)\ge 4k^2-8k-2$, by the assumption, $\sigma(\pi'')\ge 4k^2-8k-2\ge 4k^2-12k+8\ge\sigma(k-1,2k)$. Hence $\pi''$is potentially A_{k-1}-graphic, therefore π is potentially P_k-graphic. If $d_{k+1}=2k-3$, then $\pi=((2k-2)^1,d_2,\cdots,d_{k-1},(2k-3)^{k+1},d_{2k+1},1^1)$. If $d_{2k+1}\le 2k-4$, then in a similar way to the proof of Theorem 1.2, we can prove that $\pi''=(d_2-1,\cdots,d_{k-1}-1,(2k-4)^{k+1},d_{2k+1})$ is graphic. Since $\sigma(\pi'')=\sigma(\pi)-2(2k-1)\ge\sigma(k-1,2k)$, π'' is potentially A_{k-1}-graphic. Therefore π is potentially P_k-graphic. If $d_{2k+1}=2k-3$, then $\pi=((2k-2)^{2l},(2k-3)^{2k+1-2l},1^1)$ where $2\le 2l\le k-1$ because $\pi\in GS_{2k+2}$. Denote $\pi_1=((k-2)^{2l},(k-3)^{k+1-2l})$ and $\pi_2=((k-1)^{2l-2},(k-2)^{k-2l+1},(k-3)^1,0^1)$. Clearly $\sigma(\pi_1)=\sigma(\pi_2)=k^2-2k-3+2l$ and the conjugate sequence of π_1 is $\pi_1^*=((k+1)^{k-3},(2l)^1,0^3)$. It is easy to verify that π_1^* majorizes π_2. Therefore the sequence pair (π_1,π_2) is bipartite graphic(Fulkerson[8], or Brualdi and Ryser[3, Theorem 6.2.7, p178]). Obviously $((2k-2)^{2l},(2k-3)^{k+1-2l})-\pi_1=(k^{k+1})$ is the degree sequence of the complete graph K_{k+1} of order $k+1$. Denote $\pi_3=((2k-3)^k,1^1)-\pi_2=(k^1,(k-1)^{k+1-2l},(k-2)^{2l-2},1^1)$. By the off-diagonal leftmost matrix $\overline{A(\pi_3)}$ of π_3, we obtained $\overline{\pi_3}=(k^1,(k-1)^{k-2},(k-2)^1,(k+2-2l)^1,1^1)$, so $\sigma_t(\pi_3)\le\sigma_t(\overline{\pi}_3)$ for $t=1,2,\cdots,k+1$. By Theorem B, π_3 is graphic. Thus π is potentially P_k-graphic. □

Theorem 2.3. $\sigma(5,12)\le 80$ and $\sigma(5,n)\le 8n-18$ for $n\ge 13$.

Proof. By Lemma 2.2, $\sigma(5,12)\le 4\times 5^2-4\times 5=80$.

We are going to use induction on $n \geq 13$ to prove that, if $\pi = (d_1, d_2, \cdots, d_n) \in GS_n$ has no zero terms and $\sigma(\pi) \geq 8n - 18$, then π is potentially P_k-graphic. If $d_n \leq 3$, then by Theorem C, $\pi' = (d_1 - 1, \cdots, d_{d_n} - 1, d_{d_n+1}, \cdots, d_{n-1}) \in GS_{n-1}$ without zero terms and $\sigma(\pi') = \sigma(\pi) - 2d_n \geq 8n - 18 - 2 \times 3$. Clearly, if $n = 13$,then $\sigma(\pi') \geq 80 \geq \sigma(5, 12)$. If $n > 13$, then $\sigma(\pi') \geq 8(n-1) - 18 \geq \sigma(5, n-1)$. Therefore both π' and π are potentially P_5-graphic. If $d_n \geq 4$, then by Theorem C, $\pi' = (d_2 - 1, \cdots, d_{d_1+1} - 1, d_{d_1+2}, \cdots, d_n) \in GS_{n-1}$ without zero terms and $\sigma(\pi') \geq \sigma(\pi) - 2d_1 \geq 8n - 18 - 2(n-1) = 6n - 16 = 6(n-1) - 10$. By Theorem 0.3, $\sigma(4, n-1) = 6(n-1) - 10$ for $n - 1 \geq 13$, therefore π' is potentially A_4-graphic. If $d_1 = n - 1$, or there exists t, $6 \leq t \leq d_1 + 1$, such that $d_t > d_{t+1}$, then π is potentially P_5-graphic. Hence we may assume $n - 2 \geq d_1 \geq \cdots \geq d_5 \geq d_6 = \cdots = d_{d_1+2} \geq \cdots \geq d_n \geq 4$. If $d_5 > d_6$, then by Theorem C, the residual sequence $\pi'' = (d_1 - 1, \cdots, d_l - 1, d_{l+1}, \cdots, d_{d_1}, d_{d_1+2}, \cdots, d_n)$ obtained by laying off $d_{d_1+1} = l$ from π belongs to GS_{n-1}, has no zero terms and $\sigma(\pi'') = \sigma(\pi) - 2d_{d_1} \geq 6(n-1) - 10 = \sigma(4, n-1)$, where $n - 1 \geq 13$, therefore π'' is potentially A_4-graphic. Hence π is potentially P_5-graphic. So we may further assume $n - 2 \geq d_1 \geq \cdots \geq d_5 = d_6 = \cdots = d_{d_1+2} \geq \cdots \geq d_n \geq 4$. By the remark of Lemma 2.1, $d_6 \geq 5$. By Theorem 1.1, π is potentially P_5-graphic. □

Theorem 2.4. For $k \geq 5$

$$\sigma(k, n) \leq \begin{cases} 2n(k-2) + 8, & if\ 2k + 2 \leq n \leq \binom{k}{2} + 3 \\ (k-1)(2n-k) + 2, & if\ n \geq \binom{k}{2} + 3. \end{cases} \tag{1}$$

Proof. We use induction on $k \geq 5$. If $k = 5$, then $2k + 2 = 12$, $\binom{k}{2}+3 = 13$ and

$$2 \times 12 \times (5-2) + 8 = 80$$

$$2 \times 13 \times (5-2) + 8 = 86 = (5-1) \times (2 \times 13 - 5) + 2.$$

Hence, (1) holds for $k = 5$ by Theorem 2.3. Suppose (1) holds for $k - 1 \geq 5$, i.e. ,

$$\sigma(k-1, n) \leq \begin{cases} 2n(k-3) + 8, & if\ 2k \leq n \leq \binom{k-1}{2} + 3, \\ (k-2)(2n-k+1) + 2, & if\ n \geq \binom{k-1}{2} + 3. \end{cases} \tag{2}$$

Now we prove (1) holds for $k > 5$ by induction on $n \geq 2k + 2$. If $n = 2k + 2$,then by (2), $\sigma(k-1, 2k) \leq 4k^2 - 12k + 8$ and $\sigma(k-1, 2k+1) \leq 4k^2 - 10k + 2$. By Lemma 2.2, if $\pi = (d_1, d_2, \cdots, d_{2k+2}) \in GS_{2k+2}$, has no zero terms and $\sigma(\pi) \geq 4k^2 - 4k = 2(2k + 2)(k-2) + 8$,then π is potentially P_k-graphic, i.e., $\sigma(k, 2k+2) \leq 2(2k+2)(k-2) + 8$, therefore (1)holds for $n = 2k + 2$.

Suppose (1) holds for $n - 1 \geq 2k + 2$, i.e.

$$\sigma(k,n-1) \le \begin{cases} 2(n-1)(k-2)+8, & if\ 2k+2 \le n-1 \le \binom{k}{2}+3, \\ (k-1)(2n-k-2)+2, & if\ n-1 \ge \binom{k}{2}+3. \end{cases} \quad (3)$$

Now we will prove (1) holds for $n \ge 2k+3$. We only need to prove that if $\pi = (d_1, d_2, \cdots, d_n) \in GS_n$ without zero terms and

$$\sigma(\pi) \ge \begin{cases} 2n(k-2)+8, & if\ 2k+2 < n \le \binom{k}{2}+3, \\ (k-1)(2n-k)+2, & if\ n \ge \binom{k}{2}+3, \end{cases} \quad (4)$$

then π is potentially P_k-graphic.We consider two cases as follows.

Case 1: $d_n \le k-2$.Denote $\pi' = (d_1-1, \cdots, d_{d_n}-1, d_{d_n+1}, \cdots, d_{n-1})$.Then by Theorem C,$\pi' \in GS_{n-1}$,has no zero terms and $\sigma(\pi') = \sigma(\pi) - 2d_n \ge \sigma(\pi) - 2(k-2)$. If $2k+2 < n \le \binom{k}{2}+3$,then $\sigma(\pi') \ge \sigma(\pi) - 2(k-2) \ge 2(n-1)(k-2)+8$.By (3),we obtain $\sigma(\pi') \ge \sigma(k, n-1)$. If $n \ge \binom{k}{2}+3$,then

$$\begin{aligned} \sigma(\pi') &\ge \sigma(\pi) - 2(k-2) \ge (k-1)(2n-k)+2-2(k-2) \\ &= (k-1)(2(n-1)-k)+2+2. \end{aligned}$$

By (3), $\sigma(\pi') \ge \sigma(k, n-1)$, therefore π' is potentially P_k-graphic. Hence π is potentially P_k-graphic.

Case 2: $d_n \ge k-1$. Denote $\pi' = (d_2-1, \cdots, d_{d_1+1}-1, d_{d_1+2}, \cdots, d_n)$. By Theorem C, $\pi' \in GS_{n-1}$, has no zero terms and $\sigma(\pi') = \sigma(\pi) - 2d_1$. If $2k+2 < n \le \binom{k}{2}+3$, then by the assumption (4), $\sigma(\pi') \ge 2n(k-2)+8-2(n-1) = 2n(k-3)+10$. Notice that $\binom{k}{2}+2 = \binom{k-1}{2}+k+1$, therefore $2k+2 < n \le \binom{k}{2}+3$ implies $2k+2 \le n-1 \le \binom{k-1}{2}+k+1$. If $2k+2 \le n-1 \le \binom{k-1}{2}+3$,then by (2)

$$\begin{aligned} \sigma(\pi') &\ge 2n(k-3)+10 = 2(n-1)(k-3)+2(k+2) \\ &> 2(n-1)(k-3)+8 \ge \sigma(k-1, n-1). \end{aligned}$$

If $\binom{k-1}{2}+3 \le n-1 \le \binom{k-1}{2}+k+1$,then by (2)

$$\begin{aligned} \sigma(\pi') &\ge 2n(k-3)+10 = (2(n-1)-(k-1))(k-2)+2+2(\tbinom{k-1}{2}+k+1-(n-1)) \\ &\ge \sigma(k-1, n-1). \end{aligned}$$

If $n \ge \binom{k}{2}+3$,then by the assumption (4)

$$\sigma(\pi') \ge \sigma(\pi) - 2d_1 \ge (k-1)(2n-k)+2-2(n-1) = (k-2)(2(n-1)-(k-1))+2.$$

Because $n-1 \ge \binom{k}{2}+2 = \binom{k-1}{2}+k+1 \ge \binom{k-1}{2}+3$, we have $\sigma(\pi') \ge \sigma(k-1, n-1)$ by (2). Hence π' is potentially P_{k-1}-graphic.

If $d_1 = n-1$, or there exists t, $k+1 \le t \le d_1+1$, such that $d_t > d_{t+1}$, then π is potentially P_k-graphic. Hence we may assume

$$n-2 \ge d_1 \ge \cdots \ge d_k \ge d_{k+1} = \cdots = d_{d_1+2} \ge \cdots \ge d_n \ge k-1.$$

If $d_k > d_{k+1}$, then $\pi'' = (d_1-1, \cdots, d_l-1, d_{l+1}, \cdots, d_{d_1}, d_{d_1+2}, \cdots, d_n) \in GS_{n-1}$, where $l = d_{d_1+1}$, and π'' has no zero terms. In a similar way as above argument, we may prove $\sigma(\pi'') \geq \sigma(k-1, n-1)$, i.e., π'' is potentially P_{k-1}-graphic. Therefore π is potentially P_k-graphic. So we may further assume

$$n-2 \geq d_1 \geq \cdots \geq d_k = d_{k+1} = \cdots = d_{d_1+2} \geq \cdots \geq d_n \geq k-1,$$

where $n \geq 2k+3$. By the remark of Lemma 2.1, $d_{k+1} \geq k$. It follows from Theorem 1.1 that π is potentially P_k-graphic. $\square$

The following is our main result.

Theorem 2.5. If $k \geq 5$, then $\sigma(k,n) = (2n-k)(k-1)+2$ for $n \geq \binom{k}{2}+3$.

Proof. This is an immediate consequence of Theorem 2.4 and the lower bound of $\sigma(k,n)$. $\square$

Corollary 2.6. $\sigma(k, 2k+2) = 4k^2 - 4k$ for $k \geq 5$.

Proof. By Theorem 2.4, $\sigma(k, 2k+2) \leq 4k^2 - 4k$ for $k \geq 5$. Now consider the sequence $\pi = ((2k-3)^{2k+1}, 1^1)$. Clearly, $\sigma(\pi)$ is even and $f(\pi) = 2k-3$. It is easy to obtain $\overline{\pi} = ((2k+1)^1, (2k)^{2k-4}, (2k-3)^1, 0^4)$ from its off-diagonal leftmost matrix $\overline{A(\pi)}$. Hence, $\sigma_i(\pi) \leq \sigma_i(\overline{\pi})$ for $1 \leq i \leq 2k-3$. By Theorem B, $\pi \in GS_{2k+2}$.

Assume that G is one of the realizations of π and the subgraph induced by the vertex subset $U = \{u_1, \cdots, u_{k+1}\}$ of G is complete. Then the edge number of G from U to $V(G)\backslash U$ is $r = (k+1)(k-3)$. Therefore the edge number of the induced subgraph $G[V(G)\backslash U]$ of G is $\sigma(\pi)/2 - r - k(k+1)/2 = \binom{k}{2}+2$. Because $d_{2k+2} = 1$, the edge number of $G[V(G)\backslash U]$ is at most $\binom{k}{2}+1$, a contradiction. Hence, π is not potentially P_k-graphic, therefore $\sigma(k, 2k+2) \geq \sigma(\pi)+2 = 4k^2-4k$. Thus $\sigma(k, 2k+2) = 4k^2-4k$. $\square$

Remark: By Theorem 0.3, $\sigma(4, 10) = 6 \times 10 - 10 = 50 > 48 = 4 \times 4^2 - 4 \times 4$, so Corollary 2.6 does not hold for $k \leq 4$.

3 Conclusion

For any given integer number $k \geq 2$, denote by $N(k)$ the smallest positive integer number m such that for any integer number $n \geq m$, $\sigma(k,n) = (k-1)(2n-k)+2$. It follows by Theorems A, 0.1, 0.2 and 0.3, that $N(2) = 6$, $N(3) = 8$ and $N(4) = 10$. By Theorem 2.5 and Corollary 2.6, $N(k)$ exists for $k \geq 5$ and $2k+3 \leq N(k) \leq \binom{k}{2}+3$. In particular, if $k = 5$, $13 = 2 \times 5 + 3 \leq N(5) \leq \binom{5}{2} + 3 = 13$. So $N(5) = 13$. As to the problem on how to determine the exact value of $N(k)$ for $k \geq 6$, it needs further study.

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