

Partial Result on Hadwiger's Conjecture

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Abstract

Let $D = (d_1, d_2, \dots, d_n)$ be a graphic sequence with $0 \leq d_1 \leq d_2 \leq \dots \leq d_n$. Any simple graph G with D its degree sequence is called a realization of D . Let $R[D]$ denote the set of all realizations of D . We say that D is H -free if no graph in $R[D]$ contains H as an induced subgraph. In this paper, we prove that Hadwiger's Conjecture is true for graphs whose degree sequences are claw-free or $\overline{K}_{1,3}$ -free.

Key Words : Graph Minor, Claw-free, Degree Sequences

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1 Introduction

All graphs in this paper are finite and have no loops or multiple edges. A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. An H *minor* is a minor isomorphic to H . For a graph G , the *Hadwiger number* $h(G)$ of G is the maximum integer k such that K_k is a minor of G . The degree sequence of a graph G is denoted by $DS(G)$ in the nonincreasing order. We say that a graph G is H -free if G does not contain H as an induced subgraph. Let $D = (d_1, d_2, \dots, d_n)$ be an integer sequence with $0 \leq d_1 \leq d_2 \leq \dots \leq d_n$. If D consists of the terms d_1, \dots, d_t having multiplicities m_1, \dots, m_t , we may write $D = (d_1^{m_1}, \dots, d_t^{m_t})$. We say that D is *graphic* if there is a graph G with $DS(G) = D$. In those circumstances, we say that G is a *realization* of D . Let $\overline{D} = (n-1-d_n, n-1-d_{n-1}, \dots, n-1-d_1)$. Then D is graphic if and only if \overline{D} is graphic. For a graphic sequence D , let $R[D]$ denote the set of all realizations of D . We say that D is H -free if every realization of D is H -free. Note that if D is H -free, then \overline{D} is \overline{H} -free, where

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\overline{H} is the complement of H . A graph is called a *claw* if it is isomorphic to $K_{1,3}$. As usual we denote by $\chi(G)$ the chromatic number of G and by $\alpha(G)$ the independence number of G .

In this paper, we prove the following

Theorem 1.1 *Let G be a graph. If $DS(G)$ is claw-free or $\overline{K}_{1,3}$ -free, then $h(G) \geq \chi(G)$.*

Our motivation was threefold. First, Theorem 1.1 gives a partial result on Hadwiger's Conjecture [8], the following.

Conjecture 1.2 *For every integer $t \geq 1$, every graph with no K_{t+1} minor is t -colorable.*

Hadwiger's conjecture is trivially true for $t \leq 2$, and reasonably easy for $t = 3$, as shown by Dirac [7]. However, for $t \geq 4$, Hadwiger's conjecture implies the Four Color Theorem [1, 2, 9]. (To see that, let H be a planar graph, and let G be obtained from H by adding $t - 4$ vertices, each joined to every other vertex of the graph. Then G has no K_{t+1} minor, and hence is t -colorable by Hadwiger's conjecture, and hence H is 4-colorable.) Wagner [12] proved that the case $t = 4$ of Hadwiger's conjecture is, in fact, equivalent to the Four Color Theorem, and the same was shown for $t = 5$ by Robertson, Seymour, and Thomas [10]. Hadwiger's conjecture remains open for $t \geq 6$. Since Hadwiger's Conjecture is a well-known difficult problem in graph theory, and there does not seem to be a consensus in the field as to whether it "should be true", we believe that any evidence to the problem is of interest. Theorem 1.1 proves that Hadwiger's Conjecture is true for graphs whose degree sequences are claw-free or $\overline{K}_{1,3}$ -free.

In [4], Chudnovsky and Fradkin proved a weakened version of Hadwiger's Conjecture for the class of claw-free graphs. Their result states that if a graph G is claw-free, then $h(G) \geq \lceil \frac{2}{3}\chi(G) \rceil$. Note that proving Hadwiger's Conjecture for claw-free graphs is not easy. As pointed out in [4], all graphs with $\alpha(G) = 2$ are claw-free and there have been much research and multiple papers written on the subject of Hadwiger's conjecture for this class of graphs with only minimal progress [3, 5]. So the next best thing would be to prove Hadwiger's conjecture for all claw-free graphs with $\alpha(G) > 2$. However, even that seems very difficult since most of the proofs are by induction and graphs with $\alpha(G) = 2$ often appear in the base case.

The third motivating factor for studying graphs whose degree sequences are claw-free or $\overline{K}_{1,3}$ -free is related to S. B. Rao's conjecture for well-quasi-ordering of degree sequences (Note that Chudnovsky and Seymour announced at the Banff conference on Graph Minors from September 28 to October 3, 2008 that they had solved Rao's conjecture). We need to introduce more notation. Let A be a set of all graphic sequences and let $D_1, D_2 \in A$. We say that $D_1 \leq D_2$ if there exist $H \in R[D_1]$ and $G \in R[D_2]$ such that H is an induced subgraph of G . It can be easily checked that " \leq " defines a partial ordering on A and so (A, \leq) is a

poset. We say that an infinite sequence D_1, D_2, \dots in A is *good* if there exist two indices $i < j$ such that $D_i \leq D_j$; and *bad* if it is not good. S. B. Rao in 1980 conjectured that (A, \leq) is well-quasi-ordered, i.e., every infinite sequence D_1, D_2, \dots in A is good. Note that if D_1, D_2, \dots is a counterexample to Rao's Conjecture, then $D_1 \not\leq D_i$ for all $i \geq 2$. Let L be a fixed graph such that $D_1 \leq DS(L)$. Then the infinite sequence $D(L), D_2, D_3, \dots$ is also bad. Thus D_i is L -free for all $i \geq 2$. So to prove Rao's Conjecture it suffices to prove the following:

For a fixed graph L , every infinite L -free sequence is good.

It is easy to see that any graph of order k can be realized from k disjoint copies of k -claws $K_{1,k}$ by using the edge exchange method, which will be introduced later. This leads us to study claw-free graphic sequences (see Theorem 2.1 below). We hope to be able to solve Rao's conjecture by using this method. It seems that excluding k disjoint copies of k -claws is very difficult. In this paper, we will use Theorem 2.1 to prove that Hadwiger's Conjecture is true for graphs whose degree sequences are claw-free or $\overline{K}_{1,3}$ -free.

Note that Zverovich [13] proved that A is well-quasi-ordered by the subgraph relation. This is false, as subgraph relation is not transitive. Note that K_4 is a subgraph of $K_4 \cup K_2$, and a realization of $DS(K_4 \cup K_2)$ is a subgraph of $K_6 \setminus M$, where M is a perfect matching of K_6 . But $DS(K_6 \setminus M)$ is K_4 -free.

We need to introduce more notation. Let G be a graph. The minimum degree and maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The *complement* of G is denoted by \overline{G} . If $X \subseteq V(G)$, we denote the subgraph of G induced on X by $G[X]$. We use $G \setminus X$ denote the subgraph of G induced on $V(G) - X$. By $N(x)$ we denote the set of vertices of G that are adjacent to x . By abusing notation we will denote by $N(x)$ the graph induced by the set $N(x)$. We define $N[x] = N(x) \cup \{x\}$, and similarly will use the same symbol for the graph induced by that set. If $A, B \subseteq V(G)$ are disjoint, we say that A is *complete* to B if every vertex in A is adjacent to every vertex in B , and A is *anticomplete* to B if A is complete to B in \overline{G} . If $A = \{x\}$, we simply say x is complete to B or x is anticomplete to B . We use $G[A, B]$ to denote the bipartite graph obtained from the induced subgraph $G[A \cup B]$ by deleting all edges with both ends in A or in B . If H, J are two disjoint subgraphs of G , by abusing notation we will say that H is complete (resp. anticomplete) to J if $V(H)$ is complete (resp. anticomplete) to $V(J)$. For integers $r, s, t \geq 1$. We denote by $S(r, s)$ the *double-star* obtained from two disjoint graphs $K_{1,r}$ and $K_{1,s}$ by joining the degree r vertex in $K_{1,r}$ to the degree s vertex in $K_{1,s}$. We denote by $S(r, s, t)$ the *triple-star* obtained from three pairwise disjoint graphs $K_{1,r}, K_{1,s}$ and $K_{1,t}$ by adding an edge between each pair of the degree r vertex in $K_{1,r}$, the degree s vertex in $K_{1,s}$, and the degree t vertex in $K_{1,t}$. Vertices of degree at least two in $S(r, s)$ and $S(r, s, t)$ are called *center* vertices. A complete r -partite graph with each partite of size two is denoted by $K_2(r)$, and $\overline{K}_2(r)$ is denoted by rK_2 . For two disjoint graphs G and H , we denote by $G \cup H$ and $G + H$, respectively, the *union* and

join of G and H , where $V(G \cup H) = V(G + H) = V(G) \cup V(H)$, $E(G \cup H) = E(G) \cup E(H)$, and $E(G + H) = E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$. As usual, we denote by C_n and P_n , respectively, the cycle and path on n vertices.

Let G be a realization of a graphic sequence D and let a, b, c, d be vertices in G such that ac, bd are edges in G but ab, cd are nonedges in G . Removing the edges ac, bd and replacing them with the nonedges ab, cd results in a realization H of D that may or may not be isomorphic to G . This operation is frequently referred to as an edge exchange or a 2-switch, denoted by $(a, c; b, d)$ in order (see Figure 1).

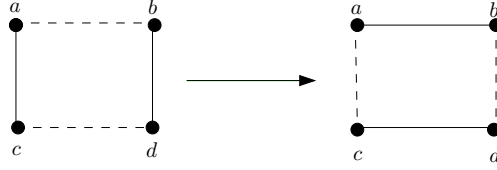


Figure 1: 2-switch

2 Main Results

In this section, we prove that Conjecture 1.2 is true for graphs whose degree sequences are claw-free or $\overline{K}_{1,3}$ -free. To do that, we need to study the realizations of those degree sequences. From now on, let $D = (d_1, d_2, \dots, d_n)$ be a graphic sequence. As mentioned earlier, D is claw-free if and only if \overline{D} is $\overline{K}_{1,3}$ -free. We will focus on characterizations of realizations of $\overline{K}_{1,3}$ -free degree sequences instead.

Theorem 2.1 *If $D = (d_1, d_2, \dots, d_n)$ is $\overline{K}_{1,3}$ -free, then for every $G \in R[D]$, either $\delta(G) = n - 3$ or G is isomorphic to one of the following graphs:*

- (1) $(C_q \cup \overline{K}_{t+1}) + K_{n-q-t-1}$, $q = 4$ or 5 ;
- (2) $(rK_2 \cup \overline{K}_{t+1}) + K_{n-2r-t-1}$ or $(r+1)K_2 + K_{n-2r-2}$;
- (3) $(K_{1,k} \cup (r+1)K_2) + K_{n-k-2r-3}$ or $(K_{1,k} \cup rK_2 \cup \overline{K}_{t+1}) + K_{n-k-2r-t-2}$;
- (4) $(K_{1,k} \cup K_{1,m} \cup rK_2 \cup \overline{K}_{t+1}) + K_{n-k-m-2r-t-3}$ or $(K_{1,k} \cup K_{1,m} \cup (r+1)K_2) + K_{n-k-m-2r-4}$;
- (5) $(S(k-1, m-1) \cup (r+1)K_2) + K_{n-k-m-2r-2}$ or $(S(k-1, m-1) \cup rK_2 \cup \overline{K}_{t+1}) + K_{n-k-m-2r-t-1}$;
- (6) $S(k-1, m-1, t+1) + K_{n-k-m-t-2}$;

- (7) $S_c(k, t + 1)$, where $S_c(k, t + 1)$ is obtained from $S(k, t + 1)$ by allowing the two center vertices having $c \geq 0$ common neighbors;
- (8) \overline{B} , where B is a graph with $V(B)$ partitioned into X, Y such that $B[X] = K_{|X|}$, $B[Y] = \overline{K}_{|Y|}$, and each vertex in X has at most one neighbor in Y ;
- (9) $L + K_{n-k-4}$, where L is obtained from C_4 by adding k pairwise nonadjacent vertices each joining to a and b , where $ab \in E(C_4)$;

where $r, t \geq 0$ and $m, k \geq 2$ are integers.

The proof of Theorem 2.1 is given in section 3. First, we would like to verify Hadwiger's Conjecture for graphs whose degree sequences are claw-free or $\overline{K}_{1,3}$ -free. We restate Theorem 1.1 as follow.

Theorem 2.2 *If D is claw-free or $\overline{K}_{1,3}$ -free, then $h(G) \geq \chi(G)$ for every $G \in R[D]$.*

Proof. It suffices to show that if D is $\overline{K}_{1,3}$ -free, then for any $G \in R[D]$, $h(G) \geq \chi(G)$ and $h(\overline{G}) \geq \chi(\overline{G})$. Assume that D is $\overline{K}_{1,3}$ -free. Let $G \in R[D]$. If G is isomorphic to one of the graphs listed in (1)-(9) in Theorem 2.1, one can see that $h(G) \geq \chi(G)$ and $h(\overline{G}) \geq \chi(\overline{G})$. So we may assume that $\delta(G) = |V(G)| - 3$. Then $\Delta(\overline{G}) = 2$ and so \overline{G} consists of disjoint cycles. Thus $h(\overline{G}) = 3 \geq \chi(\overline{G})$. It remains to show that $h(G) \geq \chi(G)$. Let T_1, T_2, \dots, T_t be all triangles in \overline{G} , where $t \geq 0$. Since $\Delta(\overline{G}) = 2$, T_1, T_2, \dots, T_t are pairwise disjoint. Let M be a maximal matching of \overline{G} and let $M' = M \cap (\bigcup_{i=1}^t E(T_i))$. Then $|M'| = t$ and by the choice of M , $G[V(G) \setminus V(M)]$ is a complete subgraph of G . Let $|M \setminus M'| = m$. Since \overline{G} consists of disjoint cycles, we may arrange the vertices of $M \setminus M'$ so that $M \setminus M' = \{x_1y_1, x_2y_2, \dots, x_my_m\}$ and $G[\{x_1, x_2, \dots, x_m\}]$ is a complete subgraph of G . One can check that $\chi(G) \leq (n - 3t - 2m) + t + m = n - m - 2t$ but $h(G) \geq (n - 3t - 2m) + t + m + \frac{t+m}{2} = n - m - 2t + \frac{t+m}{2}$. Thus $h(G) \geq \chi(G)$, as desired.

This completes the proof of Theorem 1.1. ■

3 Proof of Theorem 2.1

In this section, we will prove Theorem 2.1. Before doing so, we need some preliminary results.

Lemma 3.1 *If D is $\overline{K}_{1,3}$ -free, then for every $G \in R[D]$ and any $x \in V(G)$,*

(a) $G \setminus N[x]$ is triangle-free,

(b) $G \setminus N[x]$ is P_5 -free, and

(c) if $G \setminus N[x]$ contains a P_3 with vertices a, b, c in order, then $N(a) - b$ is complete to $N(c) - b$.

Proof. Let $G \in R[D]$ and $x \in V(G)$. Since D is $\overline{K}_{1,3}$ -free, $G \setminus N[x]$ is triangle-free. It can be easily checked that both P_5 and $K_3 \cup K_2$ are realizations of $(1^2, 2^3)$ but $K_3 \cup K_2$ contains $\overline{K}_{1,3}$ as an induced subgraph. Hence $G \setminus N[x]$ does not contain P_5 as an induced subgraph. Finally, if there exist $d \in N(a) - b$ and $e \in N(a) - b$ such that $de \notin E(G)$, then $\{a, b, c\}$ induces a triangle in $G \setminus N[x]$ after 2-switching $(a, d; c, e)$, contrary to (i). Hence $N(a) - b$ is complete to $N(c) - b$.

This completes the proof of Lemma 3.1. ■

Lemma 3.2 *If D is $\overline{K}_{1,3}$ -free, then for every $G \in R[D]$ and any $x \in V(G)$, $G \setminus N[x]$ is isomorphic to one of the following graphs:*

- (i) $rK_2 \cup \overline{K}_t$
- (ii) $K_{1,k} \cup rK_2 \cup \overline{K}_t$
- (iii) $K_{1,k} \cup K_{1,m} \cup rK_2 \cup \overline{K}_t$
- (iv) $C_q \cup \overline{K}_t$, where $q = 4$ or 5
- (v) $S(k-1, m-1) \cup rK_2 \cup \overline{K}_t$

where $r, t \geq 0$ and $k, m \geq 2$ are integers.

Proof. Let $G \in R[D]$ and $x \in V(G)$. Let $H = G \setminus N[x]$. If $\Delta(H) \leq 1$, then H is isomorphic to $rK_2 \cup \overline{K}_t$ for some integers $t, r \geq 0$, as desired. So we may assume that $\Delta(H) \geq 2$. By Lemma 3.1, $DS(H)$ is triangle-free and P_5 -free. Thus for any $v \in V(H)$, $N_H(v)$ is an independent set, and so any connected component of size at least three in H contains an induced P_3 . If H contains more than three connected components of size at least three, by performing 2-switch twice, one can obtain a realization of $DS(H)$ which contains an induced P_5 , contrary to the fact that $DS(H)$ is P_5 -free. Hence H contains at most two connected components of size at least three. Since $\Delta(H) \geq 2$, H contains at least one connected component of size at least three. Let $y \in V(H)$ with $d_H(y) = \Delta(H)$. Let $k = d_H(y)$. Then $k \geq 2$ and $|N_H[y]| \geq 3$. Assume that $d(z) = 1$ for any $z \in N(y)$. Then $N_H[y]$ is isomorphic to $K_{1,k}$. If $N_H[y]$ is the only component of size at least three in H , then $H \setminus N_H[y]$ consists of disjoint K_2 's and K_1 's. Thus H is isomorphic to $K_{1,k} \cup rK_2 \cup \overline{K}_t$ for some integers $t, r \geq 0$, as desired. Hence we may assume that H has another component, say J , of size at least three. Clearly, J is isomorphic to $K_{1,m}$ for some integer $m \geq 2$. Note that $H - N_H[y] - J$ consists of disjoint K_2 's and K_1 's. Thus H is isomorphic to $K_{1,k} \cup K_{1,m} \cup rK_2 \cup \overline{K}_t$ for some integers $t, r \geq 0$.

So we may assume that y has at least one neighbor of degree at least two in H . Then $N_H[y]$ is the only component of size at least three in H , otherwise we would obtain a P_5 after

one 2-switch if there is another component of size at least three. Thus $H \setminus N_H[y]$ consists of disjoint K_2 's and K_1 's. Assume that y has at least two neighbors, say z, w , with $d_H(z) \geq 2$ and $d_H(w) \geq 2$. Since H is triangle-free and P_5 -free, y has exactly two neighbors z, w in H . By Lemma 3.1(c), $N_H[y]$ is isomorphic to C_4 or C_5 . Then $H \setminus N_H[y]$ contains no K_2 , otherwise we would obtain a P_5 after one 2-switch. Thus H is isomorphic to $C_4 \cup \overline{K}_{|H|-4}$ or $C_5 \cup \overline{K}_{|H|-5}$. What remains to consider is that y contains exactly one neighbor, say u , with $d_H(u) \geq 2$. Then, for any $a \in N_H(u) - y$, $d_H(a) = 1$. Thus $N_H[y]$ is isomorphic to $S(|N_H(y)| - 1, |N_H(u)| - 1)$ and so H is isomorphic to $S(k - 1, m - 1) \cup rK_2 \cup \overline{K}_t$ for some integers $t, r \geq 0$ and $m = |N_H(u)| \geq 2$.

This completes the proof of Lemma 3.2. ■

We are now ready to prove Theorem 2.1.

Proof. Let x be a vertex in G with minimum degree. Let $H = G \setminus N[x]$. By the choice of x , we have

(1) v is complete to $N(x)$ for any $v \in V(H)$ with $d_H(v) = 0$.

If $d_G(x) = 0$, then by Lemma 3.2, G is isomorphic to $rK_2 \cup \overline{K}_{t+1}$, or $K_{1,k} \cup rK_2 \cup \overline{K}_{t+1}$, or $K_{1,k} \cup K_{1,m} \cup rK_2 \cup \overline{K}_{t+1}$, or $C_5 \cup \overline{K}_{t+1}$, or $C_4 \cup \overline{K}_{t+1}$, or $S(k - 1, m - 1) \cup rK_2 \cup \overline{K}_{t+1}$, where $r, t \geq 0$ and $k, m \geq 2$ are integers. So we may assume that $d_G(x) \geq 1$. Let $y \in N(x)$. By Lemma 3.2 again, we consider the following four cases.

Case 1. $H = C_q \cup \overline{K}_t$, where $q = 4$ or 5 .

Let a_1, a_2, \dots, a_q be the vertices of C_q in order. Suppose that $ya_1 \in E(G)$ but $ya_2 \notin E(G)$. Since $xa_3 \notin E(G)$, let G' be obtained from G by 2-switching $(x, y; a_3, a_2)$. Then $\{y, a_1, a_2\}$ induces a triangle in $G' \setminus N[x]$, contrary to Lemma 3.1(a). Thus y is either complete to C_q or anticomplete to C_q . We claim that y is complete to C_q . Suppose that y is anticomplete to C_q . If $q = 5$, let G' be obtained from G by 2-switching $(x, y; a_1, a_5)$. It follows that $\{a_2, a_3, a_4, a_5, y\}$ induces a P_5 in $G' \setminus N[x]$, contrary to Lemma 3.1(b). If $q = 4$, let G' be obtained from G by deleting edges xy, a_1a_2, a_1a_4 but adding edges a_1x, a_1y, a_2a_4 . Then $\{a_2, a_3, a_4\}$ induces a triangle in $G' \setminus N[x]$, contrary to Lemma 3.1(a). Hence y is complete to C_q , as claimed. By the arbitrary choice of y , it follows that $N(x)$ is complete to C_q . By (1), $V(\overline{K}_t)$ is complete to $N(x)$. Hence $N(x)$ is complete to $\{x\} \cup V(H)$ in G .

We next show that $N(x)$ induces a complete subgraph in G . Without loss of generality, suppose that there exists $z \in N(x)$ such that $yz \notin E(G)$. Let G' be obtained from G by 2-switching $(y, a_1; z, a_3)$. Then $\{a_1, a_2, a_3\}$ induces a triangle in $G' \setminus N[x]$, contrary to Lemma 3.1(a). Thus $N(x)$ is a complete subgraph in G . Therefore G is isomorphic to $(C_q \cup \overline{K}_{t+1}) + K_{|N(x)|}$. This completes the proof of Case 1.

Case 2. $H = S(k-1, m-1) \cup rK_2 \cup \overline{K_t}$ or $H = K_{1,k} \cup K_{1,m} \cup rK_2 \cup \overline{K_t}$, where $r, t \geq 0$ and $k, m \geq 2$ are integers.

We first consider the case when $H = S(k-1, m-1) \cup rK_2 \cup \overline{K_t}$. Let u, v be the two center vertices of $S(k-1, m-1)$. Let $U = N_{S(k-1, m-1)}(u) - v$ and $V = N_{S(k-1, m-1)}(v) - u$. Let $u' \in U$ and $v' \in V$. Assume that $N(x)$ is complete to $S(k-1, m-1)$. Since $\{u, u', x'\}$ induces a triangle in G for any $x' \in N(x)$, by Lemma 3.1(a), $rK_2 \cup \overline{K_t}$ is complete to $N(x)$. If there exist $z, z' \in N(x)$ such that $zz' \notin E(G)$, then a triangle $\{u, v, u'\}$ can be obtained in $G \setminus N[x]$ after 2-switching $(z, u'; z', v)$, contrary to Lemma 3.1. Hence $N(x)$ induces a complete subgraph in G . One can see that G is isomorphic to $(S(k-1, m-1) \cup rK_2 \cup \overline{K_{t+1}}) + K_{|N(x)|}$. So we may assume that $N(x)$ is not complete to $S(k-1, m-1)$. Without loss of generality, we may assume that

(2) y is not complete to $S(k-1, m-1)$.

Assume that $yu \notin E(G)$. We claim that y is anticomplete to $S(k-1, m-1)$. If $yv \in E(G)$, then $\{y, u, v\}$ induces a triangle in $G \setminus N[x]$ after 2-switching $(u, u'; y, x)$, a contradiction. Thus $yv \notin E(G)$. If $yu' \in E(G)$, then $\{y, u, u'\}$ induces a triangle in $G \setminus N[x]$ after 2-switching $(u, v; y, x)$, a contradiction. Thus $yu' \notin E(G)$ and similarly, y is anticomplete to U . By symmetry of u and v , y is anticomplete to V . Thus y is anticomplete to $S(k-1, m-1)$, as claimed. We next claim that y is anticomplete to $\overline{K_t} \cup rK_2$. Suppose that there exists $a \in V(\overline{K_t} \cup rK_2)$ such that $ay \in E(G)$. Then $\{a, y, u, v, v'\}$ induces a P_5 in $G \setminus N[x]$ after 2-switching $(u', u; x, y)$, contrary to Lemma 3.1. Thus y is anticomplete to $\overline{K_t} \cup rK_2$, as claimed. By (1), $t = 0$. So y is anticomplete to rK_2 . By the choice of x , $N[y] = N[x]$ and $rK_2 \cup U \cup V$ is complete to $N(x) - y$. Note that for any $z \in N(x) - y$, $\{x, y, z\}$ induces a triangle in G . By Lemma 3.1, $\{u, v\}$ is complete to $N(x) - y$. Hence $S(k-1, m-1) \cup rK_2$ is complete to $N(x) - y$. One can easily check that $N(x)$ induces a complete subgraph in G . Hence G is isomorphic to $S(k-1, m-1) \cup (r+1)K_2 + K_{|N(x)-y|}$.

So we may assume that $yu \in E(G)$. By symmetry of u and v , we may further assume that $yv \in E(G)$. Now by the arbitrary choice of y , we may assume that $\{u, v\}$ is complete to $N(x)$. Note that for any $x' \in N(x)$, $\{x', u, v\}$ induces a triangle in G . By Lemma 3.1,

(3) $N(x)$ is complete to $rK_2 \cup \overline{K_t}$.

By (2), y is not complete to $S(k-1, m-1)$. Let $z \in U \cup V$ be such that $yz \notin E(G)$. By symmetry of u and v , we may assume that $z \in U$. By Lemma 3.1(a), $G \setminus N[z]$ is triangle-free and so y is anticomplete to V . Similarly, y is anticomplete to U . Hence y is anticomplete to $U \cup V$. By the choice of x , $U \cup V$ must be complete to $N(x) - y$. Thus $N(x) - y$ is a complete subgraph. If there exists $w \in N(x)$ such that $yw \notin E(G)$, then $\{u, v, v'\}$ induces a triangle in $G \setminus N[x]$ after 2-switching $(y, u; w, v')$, a contradiction. Thus $N(x)$ is a complete subgraph. We next show that $r = 0$. Suppose that $r > 0$. Let $ab \in E(rK_2)$. By (3), y is complete to

rK_2 . But then $\{a, b, u, v, v'\}$ induces a P_5 in $G \setminus N[x]$ after 2-switching $\{u, u'; b, y\}$, contrary to Lemma 3.1(b). Thus $r = 0$, and so G is isomorphic to $S(k-1, m-1, t+1) + K_{|N(x)-y|}$ (centered at u, v, y).

We now consider the case when $H = K_{1,k} \cup K_{1,m} \cup rK_2 \cup \overline{K_t}$. Since $k, m \geq 2$, let H' be obtained from H by 2-switching $(a, a'; b, b')$, where a and b are the center vertices of $K_{1,k}$ and $K_{1,m}$, respectively, and $aa', bb' \in E(H)$. Then $H' = S(k-1, m-1) \cup (r+1)K_2 \cup \overline{K_t}$. Similar to the above argument, we can see that G is isomorphic to $(K_{1,k} \cup K_{1,m} \cup rK_2 \cup \overline{K_{t+1}}) + K_{|N(x)|}$, or $K_{1,k} \cup K_{1,m} \cup rK_2 \cup \overline{K_{t+1}}$, or $(K_{1,k} \cup K_{1,m} \cup (r+1)K_2) + K_{|N(x)-y|}$. This completes the proof of Case 2.

Case 3. $H = K_{1,k} \cup rK_2 \cup \overline{K_t}$, where $r, t \geq 0$ and $k \geq 2$ are integers.

In this case, we may assume that

(4) For any $v \in V(G)$ with $d_G(v) = \delta(G)$, $G \setminus N[v]$ is not isomorphic to $C_4 \cup \overline{K_t}$, $C_5 \cup \overline{K_t}$, $S(k-1, m-1) \cup rK_2 \cup \overline{K_t}$, and $K_{1,k} \cup K_{1,m} \cup rK_2 \cup \overline{K_t}$, where $r, t \geq 0$ and $k, m \geq 2$ are integers.

Let u be the vertex of degree k in $K_{1,k}$, and let $v \neq w$ be two neighbors of u in $K_{1,k}$. Assume that $yu \notin E(G)$. We claim that y is anticomplete to $K_{1,k}$. Without loss of generality, suppose that $yv \in E(G)$. Then $\{y, u, v\}$ induces a triangle in $G \setminus N[x]$ after 2-switching $(u, w; y, x)$, a contradiction. Thus y is anticomplete to $K_{1,k}$, as claimed. By the choice of x , $V(K_{1,k} - u)$ is complete to $N(x) - y$. By Lemma 3.1(c) applied to the P_3 with vertices v, u, w in order, $N(x) - y$ is a complete subgraph. We next show that y is anticomplete to $rK_2 \cup \overline{K_t}$. Suppose that there exists a vertex $a \in V(rK_2 \cup \overline{K_t})$ such that $ay \in E(G)$. After 2-switching $(x, y; w, u)$, $G \setminus N[x]$ contains either a double-star centered at y and u (if $a \in V(\overline{K_t})$), or an induced P_5 (if $a \in V(rK_2)$), contrary to (4) and Lemma 3.1(b). Thus y is anticomplete to $rK_2 \cup \overline{K_t}$. By (1), $t = 0$, and by the choice of x , $N[y] = N[x]$. Thus $N(x)$ is a complete subgraph. Since $G \setminus N[u]$ is triangle-free, we see that u is also complete to $N(x) - y$. Thus G is isomorphic to $(K_{1,k} \cup (r+1)K_2) + K_{|N(x)-y|}$, as desired.

We may assume that $yu \in E(G)$. By the arbitrary choice of y , we may assume that u is complete to $N(x)$. Assume that $N(x)$ is complete to $K_{1,k} - u$. By Lemma 3.1(c) applied to the P_3 with vertices v, u, w in order, $N(x)$ is a complete subgraph. Since $\{y, u, v\}$ induces a triangle in G , by Lemma 3.1(a), $rK_2 \cup \overline{K_t}$ is complete to $N(x)$. Hence G is isomorphic to $(K_{1,k} \cup rK_2 \cup \overline{K_{t+1}}) + K_{|N(x)|}$. So we may assume that $N(x)$ is not complete to $K_{1,k} - u$. Without loss of generality, we may assume that $yv \notin E(G)$. By the choice of x , v is complete to $N(x) - y$ and so $d_G(v) = \delta(G)$. Since $G \setminus N[v]$ is triangle-free and does not contain a double-star, y is anticomplete to rK_2 . If $r \geq 1$, let ab be an edge in rK_2 . Since $G \setminus N[x]$ is triangle-free, y is anticomplete to $K_{1,k} - u$. Now $G \setminus N[x]$ contains a double-star centered at u, y after switching $(x, y; a, b)$, contrary to (4). Thus $r = 0$. By (1), $N(x)$ is

complete to \overline{K}_t . If x has only one neighbor y , then G is isomorphic to $S_c(k, t + 1)$ centered at u and y , where $c = |N(y) \cap N(u)|$. So we may assume that x has at least two neighbors. Assume that y is complete to $N(x) - y$. By the choice of x , each vertex of $K_{1,k} - u$ is adjacent to at least $|N(x)| - 1$ vertices in $N(x)$. Thus $N(x)$ is a complete subgraph by Lemma 3.1(c) applied to the P_3 with vertices v, u, w in order. Let $y_1, \dots, y_s \in N(x)$ be such that each y_i has at least one non-neighbor in $K_{1,k} - u$. Clearly, the set of non-neighbors of y_i is disjoint from the set of non-neighbors of y_j , where $i \neq j$. Thus G is isomorphic to \overline{J} , where J is given in Figure 2.

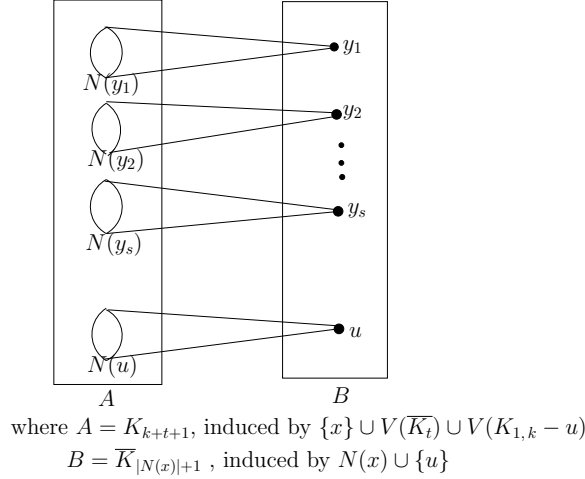


Figure 2: Graph J

We may assume that y is not complete to $N(x) - y$. Let $z \in N(x)$ be such that $yz \notin E(G)$. If $yw \in E(G)$, then $\{v, u, w\}$ induces a triangle in $G \setminus N[x]$ after 2-switching $(y, w; z, v)$, contrary to Lemma 3.1(a). Thus $yw \notin E(G)$ and so y is anticomplete to $K_{1,k} \setminus u$. By the choice of x , $K_{1,k} \setminus u$ is complete to $N(x) \setminus y$. Thus $N(x) \setminus y$ is a complete subgraph by Lemma 3.1(c) applied to the P_3 with vertices v, u, w in order. If $t > 0$, let $a \in V(\overline{K}_t)$. Then $G \setminus N[x]$, after 2-switching $(v, z; x, y)$, has a double-star (namely $G[V(\overline{K}_t) \cup \{y, u, w\}]$) centered at y and u , contrary to (4). Hence $t = 0$. Then y is complete to $N(x) - \{y, z\}$. Now G is isomorphic to $L + K_{|N(x) - \{y, z\}|}$, where L is depicted in Figure 3 and w_1, w_2, \dots, w_k are the k neighbors of u in $K_{1,k}$. This completes the proof of Case 3.

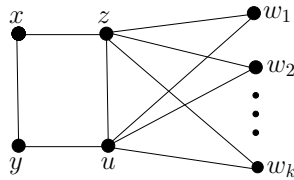


Figure 3: Graph L

Case 4. $H = rK_2 \cup \overline{K_t}$, where $r, t \geq 0$ are integers.

In this case, we may assume that

(5) For any $v \in V(G)$ with $d_G(v) = \delta(G)$, $G \setminus N[v]$ is not isomorphic to $C_4 \cup \overline{K_t}$, $C_5 \cup \overline{K_t}$, $S(k-1, m-1) \cup rK_2 \cup \overline{K_t}$, $K_{1,k} \cup K_{1,m} \cup rK_2 \cup \overline{K_t}$, and $K_{1,k} \cup rK_2 \cup \overline{K_t}$, where $r, t \geq 0$ and $k, m \geq 2$ are integers.

By (1), $\overline{K_t}$ must be complete to $N(x)$. We first consider the case when $r = 0$. Assume that $t \leq 2$. Then $\delta(G) \geq |G| - 3$, as desired. So we may assume that $t \geq 3$. Let $a, b \in V(\overline{K_t})$ with $a \neq b$. Then $d_G(a) = \delta(G)$. We claim that $N(x)$ is complete. Suppose that there exist $w, z \in N(x)$ such that $wz \notin E(G)$. Then $G \setminus N[a]$, after 2-switching $(a, w; b, z)$ in G , is isomorphic to $K_{1,t-2+1}$ (a star centered at w), contrary to (5). Thus $N(x)$ is a complete subgraph, as claimed. Hence G is isomorphic to $\overline{K_{t+1}} + K_{|N(x)|}$.

Finally, consider the case when $r > 0$. Assume that $N(x)$ is complete to rK_2 . If $N(x)$ is complete, then G is isomorphic to $(rK_2 \cup \overline{K_{t+1}}) + K_{|N(x)|}$, as desired. We may assume that $N(x)$ is not complete. Without loss of generality, we may assume that there exists $z \in N(x)$ such that $yz \notin E(G)$. If $r \geq 2$, let $ab, cd \in E(rK_2)$. Then $\{a, b, c, d\}$ induces a P_4 (a double-star) in $G \setminus N[x]$ after 2-switching $(b, y; c, z)$, contrary to (5). Thus $r = 1$. Let ab be the only edge in H . If $t \geq 1$, then $G[V(\overline{K_t}) \cup \{y, b\}]$ is an induced star in $G \setminus N[x]$ after 2-switching $(a, z; x, y)$, contrary to (5). Thus $t = 0$ and so $\delta(G) = |G| - 3$, as desired. We now consider the case when $N(x)$ is not complete to rK_2 . Without loss of generality, we may assume that y is not complete to rK_2 . Let $ab \in E(rK_2)$ be such that $ya \notin E(G)$. By the choice of x , we see that $d(a) = \delta(G)$. If $t \geq 1$, then $G[V(\overline{K_t}) \cup \{x, y\}]$ is an induced star in $G \setminus N[a]$, contrary to (5). Thus $t = 0$. Assume that $yb \in E(G)$. If there exists $c \in V(rK_2) - \{a, b\}$ such that $yc \notin E(G)$, then $\{x, y, b, a\}$ induces a P_4 (a double-star) in $G \setminus N[c]$, contrary to (5). Thus y is complete to $V(rK_2) - \{a, b\}$. Since $G \setminus N[a]$ is triangle-free, we see that $V(rK_2) = \{a, b\}$. Thus $\delta(G) = |G| - 3$ because $t = 0$. Finally, assume that $yb \notin E(G)$. Since $G \setminus N[a]$ is triangle-free and star-free by Lemma 3.1 and (5), y is anticomplete to rK_2 . It can be easily checked that $N(x)$ is a complete subgraph. We see that G is isomorphic to $(r+1)K_2 + K_{|N(x)-y|}$. This completes the proof of Case 4.

This completes the proof of Theorem 2.1. ■

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