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Note

A further extension of Yap's construction for Δ-critical graphs

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Abstract

Yap introduced in [12] (J. Graph Theory (5 (1981) 159) a method for constructing Δ-critical graphs. The construction has been extended by Plantholt [8] (J. Graph Theory 9 (1985) 371), Hilton and Zhao [5] (Discrete Appl. Math. 76 (1997) 205), and McMichael [6] (Preprint). It was shown in [5] that if G^* is a graph obtained by splitting any vertex into two vertices from a Δ-regular class 1 graph G, where $\Delta \geqslant \frac{1}{2}(\sqrt{7}-1)|G|+2$, then G^* is Δ-critical.

In this note, we further extend Yap's construction by establishing the following result: if G^* is a graph obtained from a Δ-regular class 1 graph G, where $\Delta \geqslant |G|/2$, by splitting any vertex of G into two vertices x and y such that x and y are minor vertices in G^*, then G^* is Δ-critical.

Keywords: Regular graph; Edge-coloring; Graph of class 1 or 2; Δ-critical graph; Graph obtained from a graph by splitting a vertex into two

1. Introduction

Throughout this article, all graphs we deal with are finite, simple, and undirected. We use $V(G)$, $|G|$, $E(G)$, $e(G)$, $\Delta(G)$, and $\delta(G)$ to denote, respectively, the vertex set, order, edge set, size, maximum degree, and minimum degree of a graph G. We also use O_r and O_r^t to denote, respectively, a null graph of order r and the complete t-partite graph having r vertices on each partite set. For a graph G and $x, y \in V(G)$, we write $xy \in E(G)$ if x and y are adjacent in G, and we use $N(x)$, $d(x)$, $G-x$, $G-xy$, $d_G(x,y)$, and diam(G) to denote, respectively, the neighborhood of x, degree of x, subgraph obtained from G by deleting vertex x together with its incident edges, subgraph obtained from G by deleting edge xy, distance between x and y in G, and

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the diameter of G. Let $N_G(x,y) = N(x) \cup N(y) - \{x,y\}$. Suppose $d(x) = m \geqslant 2$ and $N(x) = \{x_1, x_2, \ldots, x_m\}$. We say that G^* is a graph obtained from G by *splitting* x *into two vertices* u *and* v ($u, v \notin V(G)$) if $V(G^*) = V(G - x) \cup \{u, v\}$ and $E(G^*) = E(G - x) \cup \{uv, ux_1, \ldots, ux_r, vx_{r+1}, \ldots, vx_m\}$ for some r with $1 \leqslant r < m$. Vertices of maximum degree in G are called *major vertices* and others are called *minor vertices*. The *core* G_Δ of a graph G is the subgraph of G induced by all the major vertices of G. For a vertex v of G, we use $d_\Delta(v)$ to denote the number of major vertices of G adjacent to v in G.

An *edge-coloring* of a graph G is a map $\phi : E(G) \to C$, where C is a set of colors, such that $\phi(e) \neq \phi(f)$ whenever e and f are adjacent in G. The *chromatic index* $\chi'(G)$ of G is the least value of $|C|$ for which an edge coloring $\phi : E(G) \to C$ exists. A well-known theorem of Vizing [10] states that, for any simple graph G, $\Delta(G) \leqslant \chi'(G) \leqslant \Delta(G) + 1$. A graph G is said to be of *class* i, $i = 1, 2$, if $\chi'(G) = \Delta(G) + i - 1$. If G is a connected class 2 graph having $\Delta(G) = \Delta$ and $\chi'(G - e) < \chi'(G)$ for each edge $e \in E(G)$, then G is said to be *Δ-critical*.

Suppose π is an edge-coloring of G using $\{1, 2, \ldots, k\}$ as a color set. For any vertex v of G, let $C_\pi(v)$ denote the set of colors used to color the edges incident with v and let $C'_\pi(v) = \{1, 2, \ldots, k\} \setminus C_\pi(v)$. Clearly, π decomposes $E(G)$ into a disjoint union of color classes $E_1, E_2, \ldots, E_k$, where $E_i = \{e \in E(G) \mid \pi(e) = i\}$ for $i = 1, 2, \ldots, k$. Hence, for $i \neq j$, each connected component of $E_i \cup E_j$ is either an even cycle or an open chain. If $j \in C_\pi(v)$ and $i \notin C_\pi(v)$, then the connected component of $E_i \cup E_j$ containing v is called a $(j, i)_\pi$-chain having origin v.

A fairly long-standing problem has been the attempt to classify which graphs are of class 1, and which graphs are of class 2. The Overfull Conjecture of Chetwynd and Hilton [1], if true, would classify all graphs satisfying $\Delta(G) > \frac{1}{3}|G|$ into class 1 and class 2 graphs. A graph is called *overfull* if $e(G) > \Delta(G)\lfloor |G|/2 \rfloor$. It is easy to see that if G is overfull, then G must be of class 2. The Overfull Conjecture states:

Overfull Conjecture. If a graph G satisfies $\Delta(G) > \frac{1}{3}|G|$, then G is of class 2 if and only if G contains an overfull subgraph H with $\Delta(G) = \Delta(H)$.

There is a moderate amount of evidence to support this conjecture; for a review of this evidence, see [3] or [4]. Niessen [7] gave a polynomial algorithm to determine if a graph G satisfying $\Delta(G) \geqslant \frac{1}{2}|G|$ has an overfull subgraph H with $\Delta(H) = \Delta(G)$, and thus demonstrated that, if the Overfull Conjecture is true, then there is a polynomial algorithm to determine if a graph G satisfying $\Delta(G) \geqslant \frac{1}{2}|G|$ is of class 1 or class 2.

In [12], Yap studied the following problem. Given a Δ-regular class 1 graph G, under what circumstances is it true that any graph G^* obtained from G by splitting any vertex into two vertices is Δ-critical?

Yap [12,13] proved the following theorem.

Theorem 1.1. *Let r and t be two positive integers such that $rt \geqslant 4$ is even. Then the graph G^* obtained from $G = O_r^t$ by splitting any vertex of G into two vertices is Δ-critical, where $\Delta = \Delta(G)$.*

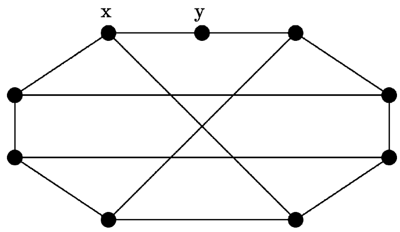

Fig. 1. G^*.

Hilton and Zhao [5] extended Yap's construction and proved the following results.

Theorem 1.2. *Let G be a connected Δ-regular bipartite multigraph with $\Delta \geqslant 2$. Let G^* be a graph obtained from G by splitting any vertex of G into two vertices. Then G^* is Δ-critical.*

Theorem 1.3. *Let G be a Δ-regular graph of even order with $\Delta \geqslant \frac{1}{2}(\sqrt{7}-1)|G|+2$. If G^* is a graph obtained from G by splitting any vertex of G into two vertices, then G^* is Δ-critical.*

Chetwynd and Hilton [2] showed that any Δ-regular graph G of even order with $\Delta \geqslant \frac{1}{2}(\sqrt{7}-1)|G|$ is of class 1. With the support of Theorems 1.1, 1.2, and 1.3, Hilton and Zhao [5] posed the following conjecture:

Conjecture A. Let G be a connected Δ-regular class 1 graph with $\Delta > |G|/3$. Let G^* be a graph obtained from G by splitting any vertex of G into two vertices. Then G^* is Δ-critical.

Hilton and Zhao [5] proved that the Overfull Conjecture implies Conjecture A. The graph G^* given in Fig. 1 is obtained from a 3-regular graph by splitting one vertex into two vertices x and y. Note that G^* is not 3-critical (see [12]). This indicates that the constant $\frac{1}{3}$ in Conjecture A cannot be reduced significantly (see [5]). In this note, we prove that if G^* is a graph obtained from a connected Δ-regular class 1 graph G, where $\Delta \geqslant |G|/2$, by splitting any vertex into two vertices x and y such that x and y are minor vertices in G^*, then G^* is Δ-critical.

2. Some useful lemmas

In this section, we state without proofs some basic results on Δ-critical graphs, which will be used in the sequel. Proofs of the first four lemmas can also be found in [13].

Lemma 2.1 (Vizing [10]). *Suppose G is a Δ-critical graph and $vw \in E(G)$. Then $d(v) + d(w) \geqslant \Delta + 2$.*

Lemma 2.2 (Vizing's Adjacency Lemma (VAL) (Vizing) [11]). *Suppose G is a Δ-critical graph and $vw \in E(G)$, where $d(v) = k$. Then*

(i) $d_\Delta(w) \geqslant \Delta - k + 1$ *if* $k < \Delta$;
(ii) $d_\Delta(w) \geqslant 2$ *if* $k = \Delta$;
(iii) $|G_\Delta| \geqslant \max\{\Delta - \delta(G) + 2, 3\}$.

Lemma 2.3 (Vizing [9]). *Let G be a class 2 graph. Then G contains a k-critical subgraph for each k, where $2 \leqslant k \leqslant \Delta(G)$.*

Lemma 2.4. *Let G be a Δ-critical graph, xy be an edge of G, and π be a Δ-coloring of $G - xy$. Then, for any $i \notin C_\pi(x)$ and $j \notin C_\pi(y)$, the $(j,i)_\pi$-chain having origin x terminates at y.*

Lemma 2.5 (Hilton and Zhao [5]). *Let G^* be a graph obtained from any connected Δ-regular class 1 graph G by splitting any vertex of G into two vertices x and y, where $\Delta \geqslant 2$. If e is an edge of G^* incident with either x or y, then $G^* - e$ is of class 1.*

3. Main results

We shall establish our main results in this section. The proofs of our main results are based on two results, namely, Lemmas 3.1 and 3.3 below. These two results are obtained by Zhang [14]. We give alternative and shorter proofs of these two lemmas here because Zhang's paper has not been published.

Lemma 3.1 (Zhang [14]). *Let G be a Δ-critical graph and xy be an edge of G, where $d(x) + d(y) = \Delta + 2$. Then any vertex in $N_G(x, y)$ is a major vertex in G.*

Proof. Suppose $w \in N_G(x, y)$ is a minor vertex in G. By symmetry of x and y, we assume that $w \in N(x)$. If $d(x) = 2$, then w is a major vertex (by VAL), which is a contradiction. Assume now that $d(x) \geqslant 3$. As $d(x) + d(y) = \Delta + 2$, it follows that y is a minor vertex in G. Since y and w are neighbors of x, we have $d(x) \geqslant d_\Delta(x) + 2 \geqslant \Delta - d(y) + 3$ (by VAL), which contradicts the fact that $d(x) + d(y) = \Delta + 2$. □

Lemma 3.2. *Let G be a Δ-critical graph and xy be an edge of G, where $d(x) + d(y) = \Delta + 2$. Let $w \in V(G) - (N(x) \cup N(y))$ and $d_G(w) < \Delta$. Then $G' = G - xy$ has a Δ-coloring such that there is a color missing at both vertices w and y.*

Proof. As G is Δ-critical, it follows that for any Δ-coloring π of $G' = G - xy$ using $S = \{1, 2, \ldots, \Delta\}$ as a color set, we have $C'_\pi(x) \cap C'_\pi(y) = \phi$. Moreover, $d_G(x) + d_G(y) = \Delta + 2$ implies that $C'_\pi(x) \cup C'_\pi(y) = S$. Hence

$$C'_\pi(x) = C_\pi(y), \quad C'_\pi(y) = C_\pi(x) \quad \text{and} \quad C'_\pi(x) \cup C'_\pi(y) = S. \tag{1}$$

Since $d_G(w) \leqslant \Delta - 1$, $C'_\pi(w) \neq \phi$. Suppose $C'_\pi(w) \cap C'_\pi(y) = \phi$. Then for any $i \notin C_\pi(y)$ and $j \notin C_\pi(w)$, we have $i \in C_\pi(w)$ and $j \in C_\pi(y)$. By (1), we have $j \notin C_\pi(x)$. By Lemma 2.4, the $(i,j)_\pi$-chain C having origin w cannot terminate at x or y. Now, by interchanging the two colors i and j in C, we obtain a Δ-coloring of G' such that color i is missing at both vertices w and y. □

Lemma 3.3 (Zhang [14]). *Let G be a Δ-critical graph and xy be an edge of G. Suppose $d(x) + d(y) = \Delta + 2$ and $d(y) < \Delta$. Then any vertex w of G at distance two from y is a major vertex in G.*

Proof. Suppose G has a minor vertex w at distance two from y. Let $u \in N(w) \cap N(y)$. By Lemma 3.1, we may assume that $u \neq x$. By Lemma 3.2, $G' = G - xy$ has a Δ-coloring π with a color set $\{1, 2, \ldots, \Delta\}$ such that there exists a color k, say, missing at both vertices w and y.

Let $\pi(yu) = i$ and $\pi(uw) = j$. Since G is Δ-critical and $d_G(x) + d_G(y) = \Delta + 2$, Eq. (1) in the proof of Lemma 3.2 will be implicitly applied here. Now by (1), $i \notin C_\pi(x)$ and $k \in C_\pi(x)$. We consider two cases separately.

Case 1: $j \notin C_\pi(y)$. In this case, $i \in C_\pi(w)$ (otherwise the $(i,j)_\pi$-chain having origin y terminates at w, which is a contradiction to Lemma 2.4). Let $z \in V(G')$ be such that $\pi(wz) = i$. Since $i \notin C_\pi(x)$ and $k \notin C_\pi(y)$, by Lemma 2.4, the $(i,k)_\pi$-chain having origin w cannot terminate at x or y. Now by interchanging the two colors of the $(i,k)_\pi$-chain having origin w, we obtain a Δ-coloring γ of G' such that $i \notin C_\gamma(w)$, $\gamma(wz) = k$, and for any $e \in E(G')$ incident with x, y or w except wz, $\gamma(e) = \pi(e)$. Note that $i \notin C_\gamma(x)$, $j \notin C_\gamma(y)$, and the $(i,j)_\gamma$-chain having origin y terminates at w, which again contradicts Lemma 2.4.

Case 2: $j \in C_\pi(y)$. We consider two subcases.

Subcase 2.1: $i \notin C_\pi(w)$. By Lemma 2.4, the $(j,k)_\pi$-chain having origin y and the $(j,k)_\pi$-chain C having origin w are vertex-disjoint. Hence, after interchanging the two colors j and k in C, we obtain a Δ-coloring γ of G' such that $i \notin C_\gamma(x)$, $k \notin C_\gamma(y)$, and the $(i,k)_\gamma$-chain having origin y terminates at w, which contradicts Lemma 2.4.

Subcase 2.2: $i \in C_\pi(w)$. Since $d_{G'}(y) \leqslant \Delta - 2$, there exists another color $l(\neq k)$ missing at y.

Suppose $l \notin C_\pi(w)$. Then by interchanging the colors of both $(j,k)_\pi$-chain and $(i,l)_\pi$-chain having origin w, we obtain a Δ-coloring γ of G' such that $k \notin C_\gamma(y)$, $i \notin C_\gamma(x)$, and the $(i,k)_\gamma$-chain having origin y terminates at w, which contradicts Lemma 2.4.

Suppose $l \in C_\pi(w)$. Let $v \in V(G')$ be such that $\pi(wv) = l$. First, by interchanging the two colors of the $(j,k)_\pi$-chain having origin w, we obtain a Δ-coloring σ of G' such that $j \notin C_\sigma(w)$, $\sigma(wu) = k$, and for any $e \in E(G')$ incident with x, y or w except wu, $\sigma(e) = \pi(e)$. Next, by interchanging the two colors of the $(l,j)_\sigma$-chain having origin w, we obtain a Δ-coloring β of G' such that $l \notin C_\beta(w)$ and for any $e \in E(G')$ incident with x, y, or w except wv, $\beta(e) = \pi(e)$. Finally, by interchanging the colors of the $(i,l)_\beta$-chain having origin w, we obtain a Δ-coloring γ of G' such that $i \notin C_\gamma(x)$, $k \notin C_\gamma(y)$, and the $(i,k)_\gamma$-chain having origin y terminates at w, which again contradicts Lemma 2.4. □

We now use Lemmas 3.1 and 3.3 to prove our main results.

Theorem 3.4. *Let G be a Δ-regular class* 1 *graph with $\Delta \geqslant 3$. Let G^* be a graph obtained from G by splitting a vertex z of G into two vertices x and y, where $d_{G^*}(y) < \Delta$. If $d_{G^*}(y,v) \leqslant 2$ for each $v \in V(G^*)$, then G^* is Δ-critical.*

Proof. Since G is of class 1, $|G|$ is even. Clearly, $e(G^*) = \Delta|G|/2 + 1$ and G^* is of class 2. We next show that for any $e \in E(G^*)$, $G^* - e$ is of class 1.

By Lemma 2.5, we may assume that e is not incident with x or y. Let $e = wv$. Suppose $G^* - e$ is of class 2. Then, by Lemma 2.3, $G^* - e$ contains a Δ-critical subgraph H. If $x \notin V(H)$ or $y \notin V(H)$, then H is a proper subgraph of G and thus H is of class 1, which is a contradiction. Hence $x, y \in V(H)$, and by Lemma 2.1, $d_H(x) + d_H(y) \geqslant \Delta + 2$. However, since x and y are the two vertices obtained by splitting z, we have $d_{G^*}(x) + d_{G^*}(y) = \Delta + 2$. Consequently, $d_H(x) + d_H(y) = \Delta + 2$, and so $N_H(x,y) = N_G(z)$. Now $d_{G^*}(y,v) \leqslant 2$ for any $v \in V(G^*)$, and $e = wv$ is not incident with x or y imply that y has a neighbor u which is also a neighbor of w in G^*. Since $d_{G^*}(y) < \Delta$, by Lemma 3.1, u is a major vertex in H. Thus $w \in V(H)$. However, by Lemma 3.3, w is also a major vertex in H, which is false. □

From the proof of Theorem 3.4, we also obtain the following results.

Theorem 3.5. *Let G be a Δ-regular class* 1 *graph with* $\mathrm{diam}(G) \leqslant 2$. *If G^* is a graph obtained from G by splitting any vertex of G into two vertices x and y such that x and y are minor vertices in G^*, then G^* is Δ-critical.*

Proof. As $\mathrm{diam}(G) \leqslant 2$, it follows that for any vertex $w \in V(G^*)$, either $d_{G^*}(w,x) \leqslant 2$ or $d_{G^*}(w,y) \leqslant 2$. Since both x and y are minor vertices in G^*, the rest of the proof of this theorem is similar to that of Theorem 3.4. □

Corollary 3.6. *Let G be a Δ-regular class* 1 *graph with $\Delta \geqslant |G|/2$. If G^* is a graph obtained from G by splitting any vertex of G into two vertices x and y such that x and y are minor vertices in G^*, then G^* is Δ-critical.*

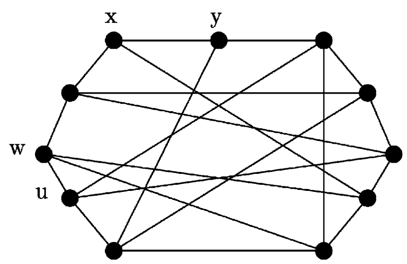

Fig. 2.

Proof. As $\Delta \geqslant |G|/2$, it follows that for any two nonadjacent vertices u and v of G, $N(u)$ and $N(v)$ has at least one vertex in common. Therefore $\text{diam}(G) \leqslant 2$. By Theorem 3.5, G^* is Δ-critical. □

Remark. The graph G^* given in Fig. 2 is obtained from a 4-regular class 1 graph G with $\text{diam}(G) = 2$ by splitting a vertex into two vertices x and y. By Theorem 3.5, G^* is 4-critical. Observe that $d_{G^*}(x, u) = 3 = d_{G^*}(y, w)$ and $\Delta = 4 < 5 = |G|/2$. This indicates that Theorem 3.5 is stronger than both Theorem 3.4 and Corollary 3.6.

Open Problem. Note that inserting a vertex into an edge is a special case of splitting a vertex into two vertices. Let G^* be a graph obtained from a connected Δ-regular class 1 graph G ($\Delta \geqslant 3$) by inserting a vertex y into any edge of G. Theorem 3.4 shows that G^* is Δ-critical when $d_{G^*}(y, v) \leqslant 2$ for any $v \in V(G^*)$, and Theorems 1.2 and 1.3 show that G^* is Δ-critical when G is a bipartite graph or when $\Delta \geqslant \frac{1}{2}(\sqrt{7} - 1)|G| + 2$. There is no further evidence to show whether or not G^* is Δ-critical when (i) G is not bipartite, (ii) $|G|/3 < \Delta < \frac{1}{2}(\sqrt{7} - 1)|G| + 2$, (iii) and $d_{G^*}(y, v) \geqslant 3$ for some $v \in V(G^*)$.

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